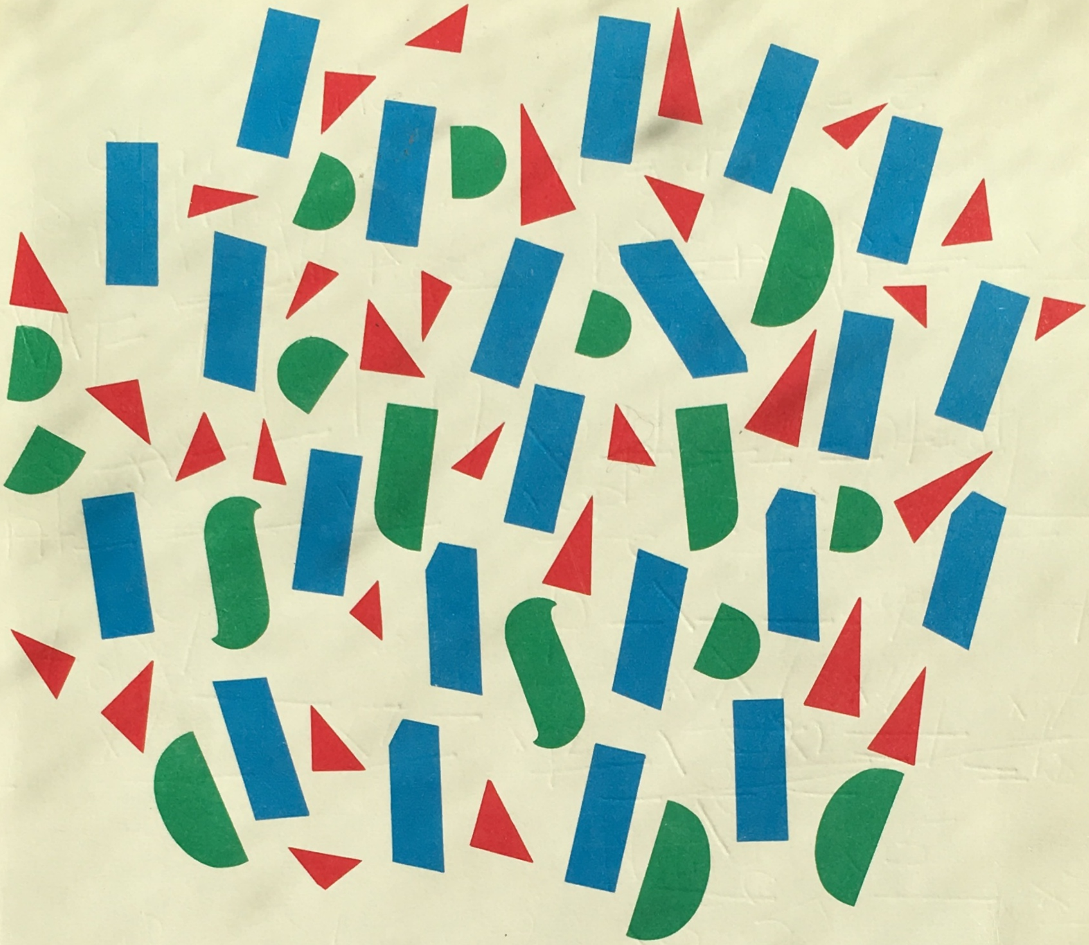
Dick Lyman



Teacher's Guide for

ATTRIBUTE GAMES AND PROBLEMS

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Elementary Science Study

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ISBN 07-018175-8

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PREFACE

The Elementary Science Study is one of many curriculum development programs in the fields of science, social studies, and mathematics under preparation at Education Development Center, Inc. EDC (a private nonprofit organization, incorporating the Institute for Educational Innovation and Educational Services Incorporated) began in 1958 to develop new ideas and methods for improving the content and process of education.

ESS has been supported primarily by grants from the National Science Foundation. Development of materials for teaching science from kindergarten through eighth grade started on a small scale in 1960. The work of the project has since involved more than a hundred scientists and educators in the conception and design of its units of study. Among these scholars have been biologists, physicists, mathematicians, engineers, and teachers experienced in working with children of all ages from kindergarten through college.

Equipment, films, and printed materials are produced with the help of staff specialists as well as the film studio, the photographic laboratory, and the production shops of EDC. At every stage of development, ideas and materials are taken into actual classrooms, where children help shape the form and content of each unit before it is released to schools everywhere.

Attribute Games and Problems

Attribute Games and Problems had its beginning in about 1953. I had been perplexed by the fact that some of the students in my fifth grade class at Shady Hill School, Cambridge, Massachusetts, who were most successful in dealing with new problems in fresh and creative ways were not particularly distinguished in traditional kinds of school work, and, conversely, other students, who were able to complete assigned work efficiently, were quite limited when dealing with problems which were not familiar to them. As I studied problem-solving approaches more closely, I became convinced that there were very important aspects of children's thinking for which we had no adequate measure. The block-sorting test which I developed to obtain more objective evidence about skills of thought became the direct ancestor of *A Blocks* and *People Pieces*.

This experience with the block-sorting test helped me become aware of the importance of classification in problem solving and the value of developing skill in handling class relationships. We experimented with many games and puzzles. One of these, which my fifth grade class helped me develop over ten years ago, is presented here in the form of *Creature Cards*.

In 1962 I had a chance to develop some of the educational implications of this work with five-year-olds. Two people were my principal collaborators in this stage of the development. David Armington, now associated with the Cleveland Public Schools, observed some of my work with children and repeated many of the activities with another class of five-year-olds. Anthony Kallet, of the Advisory Center, Leicestershire, England, participated in many of the discussions which we had and contributed new ideas of his own such as the city-planning game and the pattern games with the *Color Cubes*.

The contributions of these two people have strongly shaped the content and the style of this work. A great many suggestions came directly from the children or were suggested to us by the kinds of things which they did with the materials.

Zoltan Dienes recognized the importance of this work and gave us strong encouragement in its early stages.

The attribute materials were made available for a larger scale trial in 1964 by the Elementary Science Study. Many modifications and extensions have been made since then. Many people on the ESS staff have helped by reporting their own experiences with these materials and also their experiences in using them with children and with teachers in workshops. Madison Judson, Patricia Brinson, and Charles Ascheim, in particular, have helped me with further testing and revision. Mary Lela Sherburne and Edith H. E. Churchill have been very helpful with evaluation.

A great many teachers and student teachers, too numerous to list, have helped us by testing, criticizing, and offering alternatives. Ray Hemmings of Leicester University in England and Martin C. Michener have contributed their thinking on Latin Squares.

Elmer W. Smith did the drawings and invented many of the names and some of the ideas for *Creature Cards*.

Janet Williams, Frieda Ployer, and Adeline Naiman have provided helpful editorial assistance.

Dr. Kallet has continued to assist with development, testing, and writing. His contributions have influenced the progress of the work strongly.

William P. Hull

INTRODUCTION

Attribute Games and Problems is concerned with the development of thinking skills in children. It provides an opportunity for children to deal with problems involving classification and the relationships between classes. Experiences with such problems can help provide the familiarity and the skill necessary for solving problems in science, social studies, mathematics, or wherever classification and dealing with the relations between classes are called for.

Attribute Games and Problems includes a variety of materials: wooden blocks and cubes, plastic squares with pictures of people on them, colored loops and label cards, a set of *Creature Cards* included in a set of problem cards which suggest some of the activities which are possible, attribute stickers, and a teacher's guide for the use of the materials. Included on the cards are suggestions for games and activities simple enough for most five-year-olds and complex enough for anyone. Older students who are going to work directly with the problem cards should begin with the card *Suggested Games and Problems*.

These activities are not linked to a particular subject in the curriculum or to a narrow range of ages. The activities need not involve an entire class at the same time nor do they require an entire class period. They can be used during regularly scheduled classes in science or other subjects or more freely between classes or at other times during the school day.

Attribute Games and Problems has grown out of a long-range study of children's thinking in and out of the classroom and a continuing analysis of their difficulties and successes in a wide range of intellectual undertakings. There appear to be thinking skills and attitudes toward problem solving which are relatively independent of subject matter. Some children acquire effective thinking skills and a useful orientation toward learning in the normal course of growing up in a complex world, but the acquisition of useful skills and attitudes is by no means automatic, even under conditions which most people would regard as favorable. Many children succeed in school by repeating what they are told in a slightly different form or by pure memorization. Such strategies may be effective for meeting curriculum requirements, but they are often of little extended value. It has seemed to us that at present relatively few children develop persistence and zest for dealing with new complexities, nor do they become aware of their own intellectual power.

The results of our investigations tend to confirm a suspicion we have had that some important intellectual skills depend far more on the kinds of experiences a child has had than they do upon his age or grade in school. It is for this reason that in *Attribute Games and Problems* the same materials are used, though in different ways, from kindergarten through eighth grade and beyond.

Chapter 1 of this teacher's guide consists of reproductions of a series of cards describing games and problems, together with a commentary on the cards which suggests alternatives and extensions for use with children of different ages. These same cards are printed separately for use by students at the junior high school level or to serve as reminders of possibilities for teachers working with younger children. Perhaps the best introduction to *Attribute Games and Problems* would be to attend a workshop in which you would play some of the games suggested here, using *Color Cubes*, *People Pieces*, *A Blocks*, and *Creature Cards*. Before you use these materials with children you should have an opportunity to play some of the simple and the advanced games yourself so that you will have a sense of the range of possible activities.

Although many of the games and problems suggested in the cards can be played with younger children, do not expect them to approach the problems in just the way you have done them. Some of these cards encourage analysis which would be premature for younger children. In working from the cards you should be concerned with your own thinking, not with how you are going to present these problems to your class. It will probably be more helpful to you to return to the commentary at a later time than it will to read it when you are exploring the cards for the first time. Some of the games call for two people, so working with a partner will probably prove helpful. Some of the problems will prove challenging. You may wish to work on them over a period of several days or weeks.

If the full problem card sequence appears too formidable or if you do not have time to complete all of it before beginning to work with children, you can make a selection of the cards which would serve as a briefer form of introduction. One such selection is as follows:

A Blocks	*Color Cubes*	*People Pieces*	*Creature Cards*
1-5	1-4	1-8	1-7
11-12			
16-21			
23			
25			
27-31			

Successful work with children will require you to have in mind many ideas from which you can work, not just a few which you try out one at a time. One of the best ways of acquainting yourself with these materials is to play the games at home with other adults or with members of your family. Many people have found that they make rather good parlor games.

Please read the instructions on problem card *A Blocks 1* before opening the box containing the *A Blocks.*

Materials

You will need to order four separate items for *Attribute Games and Problems:*

1) The basic package which contains three sets of boxed materials:

A Blocks – 32 wooden blocks, all different; 6 colored cord loops; and 20 label cards and 4 blank cards

Color Cubes – 60 wooden cubes in 6 colors

People Pieces – 16 square plastic picture tiles, all different, and 8 label cards

2) The set of problem cards which includes the following:

a cover card

a card with general directions

39 problem cards for use with *A Blocks*

17 problem cards for use with *Color Cubes*

16 problem cards for use with *People Pieces*

15 *Creature Cards,* which form an additional set of materials

3) The set of gummed stickers which can represent *A Blocks* and *Color Cubes* for mapping problems and making puzzle cards

4) The teacher's guide, which includes a replica of each of the problem cards and a commentary on their use with children, and also offers background information on the learning processes involved in work with *Attribute Games and Problems*

CHAPTER ONE
COMMENTARY ON PROBLEM CARDS

When you have worked through the problems presented in the cards, you will begin to sense some of the possibilities which these materials open up. It may surprise you that some of the problems have been found appropriate for use with children as young as five. The commentary in this chapter will discuss the games and problems and indicate ways in which they may be presented to younger children. We will also discuss uses with older children, although it is our experience that almost everything we suggest for younger children applies to older ones if a teacher uses common sense and is aware of appropriate forms of presentation.

The cards in this set have been numbered simply for purposes of reference. The numbering should not be taken to mean that there is any fixed sequence for working through the games and problems: while we have suggested starting with the generation of the set, we have begun with a matrix game with equally good results. Certain cards may not prove to be particularly useful starting points, but as you work with the materials yourself, you will undoubtedly sense which activities provide good introductions. Our discussion of the cards follows the order in which they are numbered, but it is important for you to keep in mind the possibilities for rearrangement. As you work with children on a particular problem, it may occur to you, in the light of your familiarity with the entire range of games, that a useful next step might be a game which occurs well along in the series. Feel free to experiment with different orderings and encourage the children to do likewise.

In general it is best to begin with the *A Blocks* problems and games. This does not mean that all of the *A Blocks* cards need to be worked through before going on to *People Pieces, Color Cubes,* or *Creature Cards* it may be desirable to intersperse some of the latter among *A Blocks* games.

In the development of these materials we have found ourselves referring to various kinds of problems by name, but we have refrained from applying our own set of names to the cards, since names should arise after one is familiar with that which is being named. You and the children may well develop your own set of names for the various activities. There are certain broad classes of attribute activities, such as sorting, mapping, loop games, take-away games, question games, pairing games, and so forth, but these terms may not be the ones you and the children feel most at home with, and we mention them merely as suggestions.

These games can be played in many different ways: you may choose to play with an individual child or a group of children; two or more children may use the materials off in a corner; a number of children may play among themselves with you on the periphery, making suggestions or taking part as the situation seems to warrant. We have found that when an adult is playing with children it is often useful to "turn the tables," to have the children present problems for the adult to solve. Children gain skill in setting problems for one another. They become adept at inventing games and problems of their own. It may also be valuable for them to discover that they can, upon occasion, invent a problem which presents a real challenge to an adult. Another practical advantage in changing roles now and then is that doing so may give you a chance to judge how much insight a child has gained into the nature of inventing and setting problems. In making up a matrix problem (Cards 19 and 20), for example, a child may take away so many pieces that the matrix cannot be reconstructed, and this can be a point for discussion. The main value of allowing children to set problems for you, however, is that it may lead to a greater sense of partnership in a common endeavor – learning.

A Blocks 1

Do not open the box before reading this card.

Keep the box of *A Blocks* closed and hold it underneath your desk or some place out of sight. Without looking at the box, remove the cover, take out the colored loops and the cards and set them aside. Then take one of the blocks out of the box and look at it.

Can you describe it?

Can you tell what another block will look like?

Take out another block and place it next to the first one.

What else is in the box?

Take out the pieces one at a time and place them in front of you. Each time you remove one try to build up a mental picture of the possible remaining pieces. When you have taken out twelve pieces in this way, see if you can name the rest.

How many pieces do you think are left in the box?

Now look in the box.

Did you guess the right number?

Were you able to figure out what the pieces look like?

COMMENTARY: A BLOCKS 1

Most children greatly enjoy the process of "generating" the set. Older students can be given a set of problem cards and a box of *A Blocks* and allowed to proceed on their own, just as you have done. Younger children will have to be introduced to the set in another way.

The following scenario is an example of the procedure followed in classes with younger children. This method of presenting the problem can be used with an entire class or with smaller groups. No matter how big or small the group you are working with, be sure the pieces are placed in full sight of everyone as they are brought out. Proceed in a leisurely fashion, giving the children plenty of time to think about what they are seeing. In this way, even five-year-olds will be able to predict what is in the box.

Teacher:
(Holds up the closed box so everyone can see it, then shakes it a few times.) *What do you think is in this box?*

Child:
Stones?

T: *No, not stones. Listen.* (Shakes the box again.)
C: *Buttons?*
T: *No, but some of the things are shaped a little like buttons.*
C: *Balls?*
T: *No.*
C: *Candy?*
T: *No.*
C: *Wood?*
T: *Yes, all the pieces in the box are made of wood. Here's one of them.* (Puts large blue square on the table or floor where everyone can see it.) *What else do you think there is?*
C: *Do you have another one like that?*
T: *No, that's the only one just like that.*
C: *Is there a different color?*
T: *Yes. Here's a yellow square.*

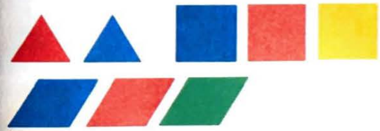
C: *Is there a red one?*
T: *Yes.* (Puts out large red triangle.)

C: (Points to the red triangle.) *I want a blue one like that.*
T: *Here's a blue triangle.*

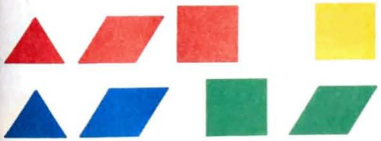
C: *Is there a different shape?*
T: *What shape would you like?* (Puts out a large blue diamond.) *What shall we call that?*

C: *A diamond. I want another diamond.*
T: *What color would you like?*
C: *Red.* (T puts it out.)
A red square. (T puts it out.)
Another diamond. (T puts out a green one.)

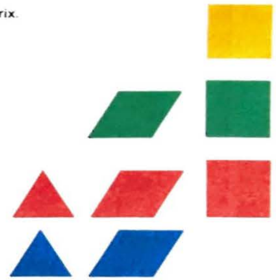
At this point children usually see that there should be a green square and a green triangle. If they run into trouble calling for pieces, you might arrange those already out according to shape



or color



or in a simple matrix.



Let the process of generating the set take its time. If a child simply asks for "a red" you can say, "What shape would you like the red to be?" Similarly, if the child asks for a certain shape, ask him for the color.

You can control the amount of information you give by the pieces which you put out. Suppose only two pieces are out – large red square and large red triangle –

and a child simply asks for "another piece." If you produce a large yellow square,

the child will know that squares come in yellow as well as in red, and may possibly infer that there is a yellow triangle in the set; if you produce a large red circle,

the child will see that there is another shape; if you bring out a small red circle,

the child will learn there is another shape *and* another size.

New information is probably best introduced gradually. It may be worth delaying the introduction of the small pieces until five or six of the large pieces are on the table. Of course, if a child specifically asks for a small piece, or for "another yellow triangle," you will be forced to bring a small piece out sooner.

Seeing the first small piece that matches a large piece already out can be exciting and may be followed by a burst of requests for small pieces to match the large pieces on the table. One reason children often find generating the set stimulating is that it reveals clearly to them their ability to think of things they have not seen. Although delighted with their skill, children will not realize that they are handling a complicated system of logical implications.

Children will often want—and should be encouraged—to follow this kind of introduction to the *A Blocks* by a considerable amount of free play on their own. Besides being fun, free play also leads to a familiarity with the set which is a prerequisite for later activities.

Young children may not be able to name the pieces immediately. Some five-year-olds may be learning the names of the shapes for the first time, although most will know the names of the colors. In naming shapes and sizes, children are apt to be quite redundant—the circle usually gets the most thorough treatment simply because there are so many good names for it. One five-year-old, struggling for a complete label, came up with "Little round small red wheel!"

At this early stage children should be allowed wide latitude in their choice of names. To avoid confusion you should eventually settle on one name for each size and shape, and use it consistently.

After generation of the set and some free exploration, playing the following game will give the children some practice in naming the *A Blocks*:

Put all the blocks out where everyone can see them. Name a piece, then ask a child to point to it. At first, children who are just learning to coordinate the three classifying attributes of each piece will find it easier to point to the piece than to supply its name. (In our trial classes we generally named the pieces in a standard order—size, color, shape—and the children usually adopted this sequence. However, "red-circle-small" is just as acceptable as "small-red-circle" and need not be corrected.)

Play this scanning game and any variations you or the children can think of. For example, let the children take turns pointing out and naming the pieces. As they play with the blocks in this way, children learn the names of the pieces quite spontaneously, often teaching each other. Some children, however, may need your support or guidance. In helping a child, it might be wise to back up and start playing the game with him from the beginning:

T: *Show me a large green square.*
C: (Points to a small green square.)
T: *That's green and it is a square, but is it the large one? Can you find the large green square?*

We feel that it is important to work with the full set of blocks. You might be tempted to simplify the process of learning to name the pieces by concentrating on just two shapes or two colors. We believe, however, that children need, and can deal with, the complexity and challenge of the entire set.

After playing these games with the children, encourage them to rehearse, repeat, and extend their experience through free play.

A Blocks 2

Take out all the pieces and see what you can do with them.

How many ways can you build with them?

Can you make an "unusual" construction?

What do you think a "usual" construction would look like?

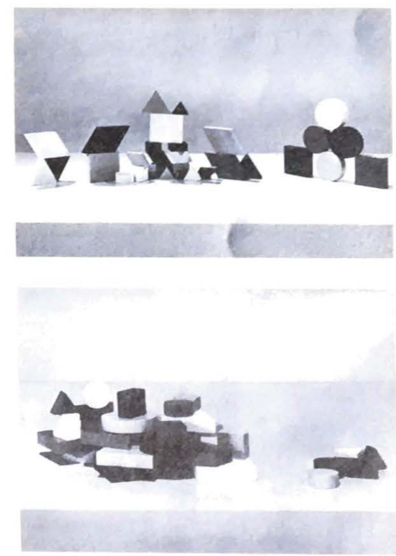
COMMENTARY: A BLOCKS 2

It is important to follow the first problem with free play and, indeed, children should be encouraged to return to free play often. There is no need to make a distinction between *work* and *play*. The *A Blocks* usually make a strong perceptual impact; the colors and shapes are attractive and invite building. If the work captures children's interest and stimulates their imagination, then it is play, and they are learning from it.

A Blocks may lend themselves to a somewhat different kind of play use than do standard kindergarten building blocks. In one way *A Blocks* may seem limited as building units, for there is only one of each kind of piece; it is not easy to construct elaborate castles because there are limited numbers of each shape out of which to fashion walls and towers. Building size may be limited, but the variety of structures which can be created is not. Each of the *A Blocks* is unique. One advantage of this uniqueness is that children may be led to explore the exciting world of asymmetry: even if a child builds symmetrically with respect to shape, he may notice that his structure does not have color symmetry. We have often noted that in their free play with *A Blocks* children begin rather conservatively with shape symmetry, and, as they grow familiar with the materials, become more audacious in exploring asymmetrical construction.

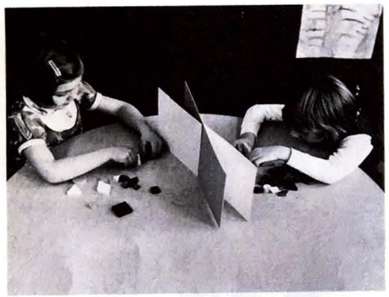
Despite the endless alternatives for building with the blocks, a child may repeat the same basic pattern, perhaps with minor variations. If he favors certain constructions, try, after a while, to interest him in making something "new," something "unusual," or something "that has never been built before."

Several children working around the same table may stimulate one another. There may be much copying, but this certainly should not be discouraged. A child may start out by imitating but he will probably end up making modifications and changes of his own.



"Unusual Constructions"

Two pieces of cardboard interlocked at right angles like this can form a serviceable screen for two or four children.



Line of sight observation.

Another way to encourage innovation in building is to suggest that two children divide a set of blocks between them and build something on opposite sides of a screen.

Children often derive considerable pleasure from building privately and then pulling the screen away to reveal their "surprise," or standing up and looking down on the other constructions.

A turntable such as a lazy-susan or an artist's modeling stand can be a most effective accessory to building activities. A large board—a three-foot square of Masonite, for example—placed on the turntable provides much more room for building and dramatizes changing relationships of the pieces as they are rotated, especially when the child's line of sight is level with the buildings. As the structures are rotated, they change continually in appearance and relationship and so become even more fascinating.

After creating interesting designs, children may begin to use the blocks as a basis for involved fantasies. They may represent buildings, people, animals, spaceships, and so forth. The liveliness of such fantasies will often be sustained if the children are left alone to play freely, without any requirement that they analyze or interpret.

As he builds, the child is, in a sense, experimenting with symbol systems, systems of representation. He needs time to devise these for himself, and to become aware of his own powers to do so. Our experience with children suggests that free building, free and imaginative use of these materials, may be related to the ability to appreciate and use such arbitrary symbol systems as language and mathematics in later education. The child who has had ample opportunity to create systems of his own may be better prepared to explore those presented to him in problem-solving situations.

A Blocks 3

Choose a shape. Take out all the pieces that have this shape even though their colors and sizes are different. Build something or arrange the pieces in some orderly way. Ask someone to remove a piece from this group or to add one to it when you are not looking.

Can you tell what piece has been taken or added?

Choose a color. Take out all the pieces of this color even though their shapes and sizes are different. Build something or arrange these pieces in an orderly way. Ask someone to take, add, or move a piece when you are not looking.

Can you tell what has been changed?

Shape and color are called *attributes* of these blocks.

COMMENTARY: A BLOCKS 3

This card introduces the students to two activities that recur throughout the problem sequence: classifying by attributes and identifying a piece added to or removed from a subset (group). Students form subsets and play take-away games to become familiar with the attributes *color*, *shape*, and *size*, as well as with different ways of classifying the pieces in the set. Those working directly from the card should work with a partner, alternating turns in the take-away games and making up variations of the suggested activities.

Younger children can be introduced to the attributes *shape* and *color* and to orderly arrangements quite naturally in a free-play situation. After two or four children have been building freely with a set of *A Blocks* for a while, you might ask them to share the pieces equally. If their method of dividing the set is random at first, see if you can get them to think of more systematic ways of sharing the pieces. Four children may do this by color, each child taking all the pieces of one color, or by shape. Two children sharing a set can each take two colors, or two shapes, or one child can take the large pieces and the other the small ones.

There are several ways of introducing the take-away game. For example, ask one child to close his eyes or to turn around while you, or another child, remove or add a piece. Another way is to slide a piece of cardboard between the child and the array of blocks in front of him and then remove or add a piece. Initially you may want to take turns playing with the children; afterwards, they can play this game (or any other they think of) with a partner.

Take-away games may be difficult at first for some children. Perhaps the easiest way to introduce them is to encourage a child who has all the pieces of one color to place each small piece on top of the corresponding large piece. When a single piece is removed from such an array, it is quite easy to identify what is missing. Two pieces at a time may then be removed. Finally the eight pieces can be scrambled and a piece removed.

A Blocks 4

Put all the large diamonds into one group and all the small diamonds into another. Make a group of the large triangles, and a group of the small triangles. Do the same thing for the circles and the squares.

You now have eight groups, four groups of large pieces and four of small pieces. (You can also say that for each shape you have a group of large pieces and a group of small pieces.)

Bring all the large pieces together into one pile, and bring all the small pieces together into another pile. The *A Blocks* set is now divided into two groups according to size.

Size, like color and shape, is an *attribute* of this set. Size, shape, and color are the three attributes we use to describe the *A Blocks*.

A Blocks 5

One attribute of this set is *shape*. Using all the blocks, make groups *(subsets)* so that each subset contains only pieces of the same shape, large and small. You will have a subset of all the triangles, a subset of diamonds, a subset of squares, and a subset of circles. Triangle, diamond, square, and circle, are called *values* of the attribute shape.

What are the values of the attribute *color?*

What are the values of the attribute *size?*

You may want to make a table like this (do not write on the card):

Attributes	Values			
Shape	?	?	?	?
Color	?			
Size	?			

COMMENTARY: A BLOCKS 4

Size is an attribute many students find troublesome in working with *A Blocks;* color and shape distinctions are clear and absolute, but size comparisons are relative. There is, in fact, a great deal of variation between pieces within the two size groups. Older (or more analytic) students may point out, for example, that two small triangles are the same size as one small diamond and, similarly, that two large triangles will just cover one large diamond.

Students may be encouraged to check the relative weights of the blocks, as we did. (In weighing one set, we found the small diamond slightly heavier than the small circle, but the large circle was heavier than the large diamond.)

With accurate measurements, your students may be able to find eight different values for the attribute *weight;* two or four are more likely. Investigations of this sort can be very useful, especially in helping older students understand that classification by three attributes represents an arbitrary decision to pay attention to certain features and not to others.

COMMENTARY: A BLOCKS 5

To communicate clearly it is necessary for us to make the distinction between *attribute* and *value*. One way of looking at the distinction between the terms is to think of them as representing different levels of abstraction. The word "color" is more abstract than the word "blue" because "color" refers to all possible colors, while "blue" refers to a limited portion of the spectrum; so in this case "color" is the attribute and "blue" is one of the values. The relationship between attribute and value is, however, strictly relative. For example, suppose one were dealing with a large group of objects of different shades of blue, but all of them blue. One might then refer to blue as an attribute of the particular group, and terms such as "sky blue," "light blue," and "navy blue" would be values. Few students will be familiar with this distinction, but those at the junior high school level may be able to grasp the implications of their experience with these materials and to extend the idea of attributes and values to other situations. By the time your students are working with *Creature Cards*, they will be inventing attributes and values for their own "creatures."

Especially with the younger children you should look for chances to use the value and attribute names informally, playing games and talking with the children:

Which color would you like?
Choose a shape . . .
Do you want the circles or the diamonds?
Which size would you prefer, large or small?

Your consistent use of the terms will help the children learn them naturally. Although they may start using these names quite rapidly, it would still be a good idea to check them occasionally and to help those who do not seem to understand.

Here are three blocks.

How are they alike? How are they different from these?

Can you think of one word that will describe this difference?

The child who is still in doubt should be shown some pairs that differ in only one attribute:

(Different in color)

(Different in shape)

(Different in size)

If the child is encouraged to find one word for each of these differences, he will soon become quite familiar with the attributes we are using.

Our title, *Attribute Games and Problems*, may prove helpful in familiarizing children with the word, whether or not they fully understand its meaning. There is no need for children to memorize or repeat any of the words used in these activities; what is important is that they feel at ease with the ideas the words imply.

A Blocks 6

Arrange the set of blocks into subsets (groups) so that each subset contains only those pieces that have the same color *and* the same shape.

How many subsets are there?

How many blocks are in each subset?

How do the pieces within a subset differ from each other?

Do you have a group of yellow diamonds?

Can you name the other groups?

COMMENTARY: A BLOCKS 6 AND 7

Like the words *attribute* and *value*, the terms *set* and *subset* are useful in defining the problem situations presented. *Set*, of course, is a common word—although it requires, in all its uses, almost two full columns in Webster's Unabridged Dictionary! Most children understand its use in expressions such as *a set of dishes* or *a set of blocks*; older children may have been exposed, through one of the new mathematics programs, to expressions such as the *set of all the children in the class* or the *set of all the animals in the United States*.

Younger children will benefit from counting the members of various subsets:

How many small reds are there?
How many large triangles?
How many red squares?
Are there more large circles or more small reds?

Questions of this sort require the children to identify members of subsets as well as to count them. While this may be difficult, it is worth spending some time on because it can lead to a greater awareness of the abstract nature of number.

Attributes and values are abstractions. We cannot point out "a large" or "a yellow" without naming the concrete object associated with the characteristic. Number is even more abstract because it is not a property of an object, but of a set of objects. In learning mathematics the study of sets, therefore, should precede the study of numbers. When classification and counting are combined, many children may grasp the abstract nature of number more easily. For this reason, the problem situations presented here may be particularly relevant to teaching and learning mathematics.

Two-Attribute Subsets

Card 6 calls for subsets of pieces alike in both color and shape. Card 7 is similar except that it explores subsets of pieces alike in color and

A Blocks 7

Arrange the blocks in subsets so that the pieces in each subset are alike in color and size.

How many subsets are there?

How many blocks are there in each subset?

How do the blocks within a subset differ from one another?

Can you think of a name for each subset?

Choose a different combination of two attributes and try to answer these questions again.

Is there another combination of two attributes you have not tried?

size, and in shape and size. The problem of grouping blocks by two common attributes may be challenging at first because it involves a sequential process, a continual shifting back and forth between two ideas. Once two blocks are matched correctly, however, the rest of the problem should become fairly straightforward.

It may help to have children focus on the one difference between pieces rather than on the two common properties. That is, it may be easier to say (or for them to think), "These are different only in size" than to say (or think), "These are alike in color and shape." Shifting back and forth between the use of positive information and the use of negative information is a valuable skill. For this reason the cards encourage students to focus on differences as well as likenesses.

Until young children have become quite familiar with the individual attributes of shape, size, and color, they will not easily be able to deal with these problems. Here are some questions you might ask to determine whether particular children are able to deal with two attributes in combination:

Can you arrange the blocks so that each subset contains only pieces of the same color?
Can you group the blocks so that the pieces in each subset are all the same shape?
Can you divide the set of blocks by size?
How many large circles are there?
How many red diamonds can you find?
How many pieces are yellow?
How many green blocks are there in this set?
How many of the pieces are triangles?
Count the number of small pieces.
Count the number of large pieces.
If John takes all the red pieces and Ann takes all the squares, who will have more pieces?
(You might follow this by asking, *Who gets the large red square?*)

A Blocks 8

Choose a value (a particular color, a shape, or a size).

Make a subset of all the pieces having this value.

Choose a value of a different attribute, and make a subset of the pieces having this value.

Take from these two subsets all the pieces that have *both* the values you have chosen.

How many pieces were in your first subset?

How many pieces were in your second subset?

How many pieces share both values?

Can you form a subset containing a different number of pieces by choosing other values?

COMMENTARY: A BLOCKS 8

This problem requires a shift in focus. The student is asked to consider combinations of values instead of combinations of attributes.

Those familiar with sets will recognize that this kind of classification results in sets comprising the *intersection* of two values. The subset of red blocks contains eight pieces. The subset of circles also contains eight pieces. The intersection of the subset of red blocks and the subset of circles contains two pieces, the red circles.

The subset of small pieces contains sixteen blocks. The subset of squares contains eight blocks. The intersection of the subset of small pieces and the subset of squares will contain four pieces, the small squares.

For younger children a questioning game similar to that suggested at the end of the commentary for Cards 6 and 7 may be helpful.

A Blocks 9

Form a subset of all the pieces which are either yellow or diamond. This subset will contain all the yellows and all the diamonds.

How many pieces are there in the subset?

Form a subset of all the pieces which are either large or red. The subset will contain all the large pieces and all the reds.

How many pieces are there in this subset?

Can you tell without using the blocks how many pieces there would be in a subset formed of pieces which are

Either triangles or green?

Either small or blue?

COMMENTARY: A BLOCKS 9

Card 8 deals with intersections. Cards 9 and 10 deal with unions. An *intersection* of the subset of diamonds and the subset of yellows contains two blocks, the yellow diamonds. The *union* of the subset of diamonds and the subset of yellows will contain all eight diamonds and all eight yellow pieces. Since two of the pieces are both diamonds and yellow, there will be fourteen pieces in the union.

Although this problem and the one on Card 10 are likely to be challenging for younger children, those who have had ample time to become acquainted with the blocks may be able to follow such suggestions as, "Suppose this box can have pieces that are *either* red *or* circle. Which ones will go in it?" It may be necessary to repeat the problem in a slightly different form before the child understands what is intended: "*Red* pieces can go in this box and *circles* can go in it too. How about this piece? Is it red? Is it a circle?"

A Blocks 10

Put into the *A Blocks* box all the pieces that are either red or circle. Put the remaining pieces aside.

Take out of the box all the pieces that are not circles.

In what way are the pieces that you have taken out alike?

Put back the pieces that are not circles and take out all the pieces that are not red.

In what way are all the pieces that you have taken out alike?

Suppose a box contains all the pieces that are either yellow or square.

What can you say about the pieces that are not square?

What can you say about the pieces that are not yellow?

Practice this kind of game until it is easy for you.

COMMENTARY: A BLOCKS 10

A piece which is *either* red *or* circle may be red, may be a circle, or it may be both red and a circle. (In everyday usage the words "either" and "or" are sometimes used to mean one *or* the other, but not both, as for example, "You may have either a lollipop or an ice cream cone." Here "either red or circle" specifically includes the piece which is both red and a circle.)

The "not-circles" which are removed from the union of the circle set and the red set are all red. The "not-red" pieces removed from the union are all circles. This idea may be confusing to students until they have repeated the game a number of times. Many younger children may be intrigued with this kind of puzzle when you present it to them, without feeling the need for analysis.

It is not especially important that students learn the terms *union* and *intersection*. It is important, however, that they have an opportunity to work with these contrasting situations.

In presenting this problem verbally it is misleading to say, "Put into the box either the red pieces or the circles," and this is an easy mistake to make. The idea which should be conveyed is that "any piece which is either red or circle can go in the box."

A Blocks 11

Can you answer the following questions without looking at the pieces?

How many red pieces are there?

How many triangles are there?

How many small pieces?

How many large circles?

How many small yellow pieces?

How many large blue diamonds?

How many green squares?

How many non-red circles?

How many non-square blues?

How many non-large triangles?

How many non-circle, non-yellows?

A Blocks 12

Group the blocks in subsets by color. Ask your partner to choose a color subset and make a building or a design with these blocks. Can you make a similar arrangement with a subset of another color?

Start again and group the blocks by size. Ask your partner to build something with the large blocks. Can you make a similar arrangement using the small blocks?

Start again and group the blocks by shape. If your partner builds something with two shapes (for example, all the squares and all the triangles) can you make a similar arrangement using the diamonds and the circles?

These games involve a kind of *map-making* in which one value stands for, or represents, another value of the same attribute.

COMMENTARY: A BLOCKS 11

Students who are working directly from the cards may write the answers to these questions on a separate piece of paper and then compare their results with the results found by others who have done the same things.

The analysis of this problem is fairly straight-forward. Since the thirty-two pieces in the set are divided into two values of size, there must be sixteen pieces of each value; that is, sixteen large pieces and sixteen small ones. There are four values of color, so there are eight pieces of each color, and the same is true of the values of shape.

Choosing a size results in a subset half the size of the original set (½ x 32). Choosing a color or shape results in a subset which is a quarter of the original set (¼ x 32). If you choose pieces which are alike in both color and shape, that is, if you combine the value of color and the value of shape, you will have a subset of two pieces (¼ x ¼ x 32 = 2). If you choose pieces which are alike in color and size, or in shape and size, there will be four pieces in each subset (¼ x ½ x 32 = 4).

Younger children, for whom such an analysis may be inappropriate, may be able to answer the questions on this card intuitively. Those who have trouble can easily count the pieces to determine the answers.

COMMENTARY: A BLOCKS 12

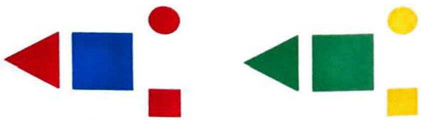
This card introduces the idea of representation, of *mapping*. Students soon learn that color mapping and size mapping are relatively simple, while mapping shapes may present a more challenging situation. Here are some good games for children in the earlier grades as well as for older children.

Color Mapping
Suggest that the children make color subsets. As many as four children can play this game with one set of *A Blocks*, each child using a different color. One child builds something with his subset and the other children try to copy it. This task requires continual shifting of attention from the building being copied to the copy and back again.

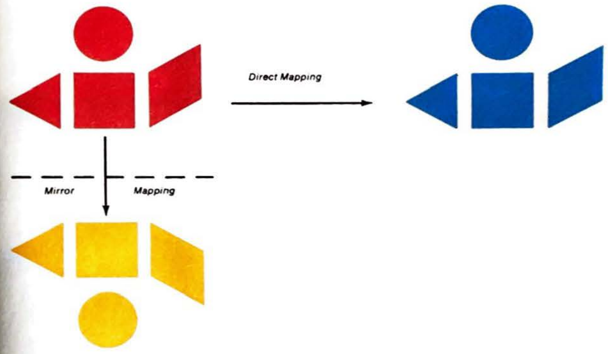
Some children may not be able to copy a completed building. You might want to vary the game so that the first child puts his blocks in place one at a time and the others follow suit, piece by piece.

Children may, at times, prefer mapping their own buildings rather than those of other children.

If two children play, they can divide the blocks so that one child has all the blocks of two colors and the second child has all the blocks of the other two colors. If the second child copies a building made by the first, paying attention only to the shape and size and using color randomly, he will have made a size and shape correspondence without making a color map.

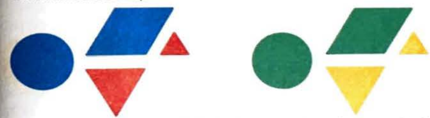


Random color usage with shape and size correspondence.



Varieties of color mapping.

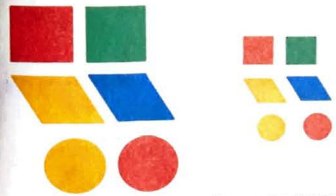
If he uses his colors systematically to represent the colors used in the building he is copying, he will have made a size and shape correspondence and a color map.



Color mapping with shape and size correspondence. (Blue is mapped by green and red is mapped by yellow.)

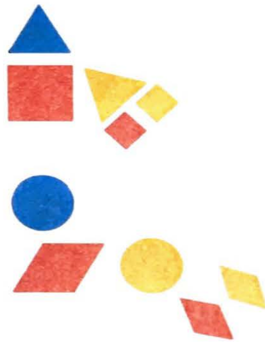
Children may produce a mirror image of the original building. If they are encouraged to explore, they may realize that there is a difference between mirror mapping and direct mapping.

Size Mapping
If one child has the small blocks and another child has the large ones, scale will be introduced into the mapping games. When the children map by size, they frequently set up color and shape correspondences: the first child builds something involving large blocks of various colors and the second child tries to duplicate it with small blocks in the same color pattern.



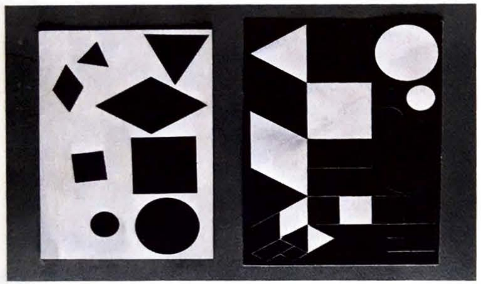
Size mapping with color and shape correspondence.

Shape Mapping
All mapping consists of setting up representational systems: a road map represents a network of roads and cities; an architect's plan represents certain features of a house; large *A Blocks* are used to represent small ones, or vice-versa; one color represents another. In shape mapping and in size mapping the representation is distorted because the form or the outline of the map differs from that of the original.



Shape mapping with size and color correspondence. (Triangles are mapped by circles and squares are mapped by diamonds.)

Children may feel a need to shift from the vertical to the horizontal plane in shape mapping. For example, if a "house" is built out of squares placed upright one upon another, a schematic representation of it can be made with circles or triangles if the representation is laid out flat on



the table. It is valuable for children to experience many variations in representation and to begin to get a feeling for how a change in value will affect the process of representation.

There is an interesting game that can be played with any one of these mappings. One person looks away while his partner exchanges one or more pairs of pieces in the construction or its representation. The first person then changes either arrangement so that the map is still correct.

Mapping With Attribute Stickers

Children who have learned how to copy a building by using one set of pieces to represent another set will be ready to use the attribute stickers to map what has been done with the *A Blocks*. The children can build a "city" with the blocks and then map what they have done by sticking the pressure-sensitive stickers onto a sheet of paper.

Another way of proceeding would be to start with a map of attribute stickers and to have the children build a city to correspond to this. They may wish, at first, to place the *A Blocks* which correspond in color, shape, and size on top of the attribute stickers. The next step will be to slide the map out from under the blocks so that the "city" and its map will be side by side.

No matter which way the city-mapping is started, you will now have a block city and a sticker city which maps it. To help children focus on the relationship between the two, it is helpful to suggest that a child "take a walk" with his fingers through the block city while another child follows each step through the map made by stickers. When children are able to see the connection between the block city and the sticker city, you can challenge them to rotate the map and continue the walk from point to point.

It may be fairly easy for children to shift attention back and forth between the block city and the sticker city when the two are oriented in the same way, but many will be confused, at first, when the map is rotated 90° or 180°. Many of us have trouble with the problem of direction in mapping. If you pose a series of problems in rapid succession, pointing to different positions in the city or on the map, many children will learn to find their way around quite easily, even though the map and the city are not oriented in the same direction.

Three sizes and six colors of shapes are supplied in the attribute stickers. Two of the sizes correspond directly with the *A Blocks*. The third size is a step smaller than the small pieces of the *A Blocks* set. Size mapping can be set up with the stickers by using the smallest stickers to stand for the small *A Blocks* pieces and the medium stickers to stand for the large *A Blocks* pieces.

If the *A Blocks* pieces are placed on edge, it will be necessary to cut the sticker material to obtain pieces which correspond to the base area of the blocks.

Puzzle Cards Made With Attribute Stickers

There are many possibilities for making puzzle cards using the attribute stickers by themselves. You may wish to make up some of these for your own students, or your children may enjoy making up problems for each other.

If you plan to use all three sizes of the attribute stickers, you will probably wish to mount them on tagboard or some form of card stock. Sheets that are 8½" x 11" would be a good size for this. These could be used by individuals or small groups of children or they could be displayed on a bulletin board. If you wish to have a smaller set, 5" x 8" filing cards could be used. In either case, the missing pieces could be changed from time to time, since the pressure-sensitive material can be reused several times. It is also possible to stick the missing piece to the back of the card so that the puzzles are self-checking.

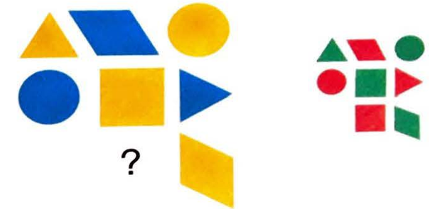
The range of difficulty is so great and the kinds of problems so varied that we are supplying suggestions and materials for you to make your own cards in collaboration with your students. There are many cards which will be appropriate for most five-year-olds. Others will be challenging for many grown-ups.

Many teachers who have played the games and worked the problems suggested in the problem cards have reported that there are so many possibilities with the *A Blocks* that they have a hard time remembering things which they would like to introduce to the children. The puzzle cards can serve as useful reminders of problems that are appropriate, as a record of the kinds of things which have been done, and as a means of assessing what children have learned and diagnosing the difficulties they may be having. We are including examples of possible puzzle cards in the places where they are related to problems discussed in the text. There is no necessary order for these. You will soon get some sense of the order of difficulty when you make up cards of your own for your students.

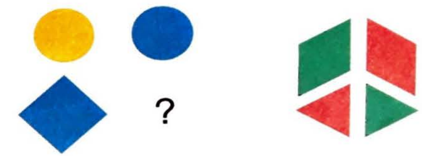
In many cases the line between a problem that is easy and one that is too difficult to solve is a very narrow one. Students may find that puzzles too difficult to solve in card form are readily solved when the same problem is presented with blocks which can be moved about. Students who solve problems easily from these indicate clearly that they know what they are doing. Those who have trouble show that they need to have the problem presented in another way, perhaps with the *A Blocks* instead of the stickers. They may need to try related problems instead, or go back to an earlier level so that they can gain skill through practice and develop confidence in what they are able to do.

Mapping With Puzzle Cards

Color-mapping, shape-mapping, or size-mapping problems all provide interesting possibilities for puzzle cards. Any of the illustrations on page 21 would serve as a good basis for a puzzle card. All that needs to be done is to copy these arrangements with the colored stickers and to leave out one or more pieces from either part of the pattern. There are a whole range of puzzle cards which would be appropriate for five-year-olds who have played mapping games with the *A Blocks*. The same type of problem becomes more challenging if two kinds of mapping are done simultaneously or if mirror mapping or rotations are included.



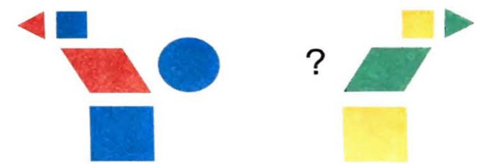
Color and size mapping, shape correspondence.



Shape and color mapping, size correspondence.

Oftentimes children will spontaneously make mirror maps.

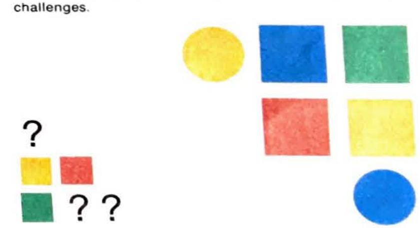
Mirror mapping: blue is mapped by yellow and red by green, but one figure is the mirror image of the other.



Color mapping – mirror.

This is a case of size mapping which has been done directly, except that one figure has been rotated.

Mirror mapping and rotations provide interesting variations and challenges.



Size mapping with rotation.

A Blocks 13

Put the blocks into the box.

Can you name all the pieces without looking at them?

If you have trouble doing this, have someone take each block out of the box and put it in front of you as you name it.

A Blocks 14

In the A Blocks set there are four colors, four shapes, and two sizes.

Before counting the blocks, can you tell how many there are altogether?

Choose two values of the attribute shape, three values of the attribute color, and one value of the attribute size.

How many pieces are there in the subset that has all the possible combinations of these values?

Here is an example. Suppose you choose the following values:

Shape	Color	Size
circle	red	large
diamond	yellow	
	blue	

In this example, the subset formed would have all the large red, yellow, and blue circles, and all the large red, yellow, and blue diamonds.

How many pieces are there in a subset that has four values of shape, three of color, and one of size? Form such a subset if you are not sure.

How many pieces will there be in a subset that has two values of shape, three of color, and two of size?

Whatever values you choose, you can find out whether a piece belongs in the subset by checking to make sure it has at least one of the values of each attribute, shape, color, and size.

COMMENTARY: A BLOCKS 13

This card presents an exercise in organizing experience, not a test of memory. Unless they have developed some ability in classifying, children who are perfectly familiar with the blocks and know their individual names may still find it difficult to recall the pieces when they are not visible. This problem gives children a chance to review and consolidate a skill they started to exercise at the very beginning of their work with the *A Blocks*.

For some children it may be helpful to repeat this exercise occasionally as the need for review becomes evident. Doing this gives the student a chance to discover his growing power to organize a considerable degree of complexity.

Children who are slow in learning the names of the pieces can be helped by a game such as this: place all the blocks on the table and ask the child to point out the pieces as you name them. For variation, allow him to remove each correctly named piece. Children may be able to take turns playing the game among themselves without you. As a piece is named by one child, his partner can place it in a separate pile.

COMMENTARY: A BLOCKS 14

This exercise can lead to the discovery that multiplying the number of values of each attribute yields the number of pieces in the subset, as explained in the commentary for Card **11**. Such insight is more apt to occur to older children, but some younger children may also find this analysis interesting.

A Blocks 15

Try to do these problems in your head.

How many subsets will you have when the A Blocks are grouped by color? (All the pieces in a subset must have the same color.) How many pieces will be in each subset?

How many subsets will there be, and how many pieces will there be in each subset, when the blocks are grouped by size?

By shape?

By size and shape?

If you are not sure of your answers, make each of the subsets and count the blocks in it. It may help if you write your answers on a piece of paper like this:

Group	Number of subsets	Number of blocks in each subset
Color		
Size		
Shape		
Shape and size		
Shape and color		
Color and size		

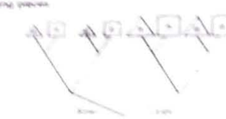
A Blocks 16

Choose two values of each attribute: two sizes, two colors, two shapes. For example, choose:

Size	Color	Shape
large	red	triangle
small	blue	square

You will have the following pieces:

large red triangle
large red square
large blue triangle
large blue square
small red triangle
small red square
small blue triangle
small blue square



Ask someone to remove one or more pieces from your subset.

Can you tell what is missing?

Make a different subset using two values of each attribute.

Can you identify one or more pieces removed from this subset?

Practice making subsets of this kind until you can do it easily.

COMMENTARY: A BLOCKS 15

This card provides a review of one-, two-, and three-attribute groupings. (This is another problem which may be more appropriate for children of ten or older children than for younger children unless they have developed a taste for analysis.) It reviews, in a different form, the ideas covered on Card 11. The commentary on Card 11 applies equally to this problem.

COMMENTARY: A BLOCKS 16 AND 17

The process of identifying pieces missing from a subset is complicated although it may be carried out swiftly with the complications appearing only when one begins to think about the task analytically. Analysis involves first the awareness that the subset is not a random one. Proceeding from this awareness one may alternate rapidly between scanning the subset, developing a conviction that "something is wrong," and focusing on individual pieces and "sub-subsets" — for example, noticing that while there are four yellow pieces there are only three red ones. Then noticing that while there are two large diamonds there is only one small one.

It is important to vary the subsets used so that children learn to rely not upon memory of specific pieces but upon the information provided by the subset, by its boundedness, its orderly, nonrandom organization. In order to help children utilize the information contained within a given subset you may want to pair the pieces or have the children pair them so that they are different in one attribute. The pairs may be alike except for size, or color, or shape. When the pieces are paired, slide a card or any other convenient screen between the blocks and the child so that he cannot see them. Remove a piece and withdraw the screen.

The screen permits a number of problems related to one subset to be presented in rapid succession. After a while, another subset can be formed.

The pairs within a subset may be formed so that there are two differences or three differences, and the take-away game repeated. In general, the single-attribute difference will permit the quickest identification of a missing piece. The three-attribute difference may be much harder for some students. When they have mastered it, you can play the take-away game without pairing the pieces at all.

The concept of pairing may develop slowly in young children. To make the idea clearer, suggest that they think of things which customarily

A Blocks 17

Choose two values of color and two values of shape. Take out all the pieces that have these colors and these shapes. Put any two of the eight pieces together to make a pair. Pair the remaining six pieces to match the first two.

For example, if the pieces in the first pair are alike except for color, the pieces of the other pairs must also be alike except for color. Each of these pairs will have one difference in common – color.

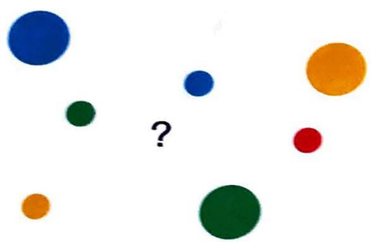
Can you make pairs that have two common differences?

Can you make pairs that have three common differences?

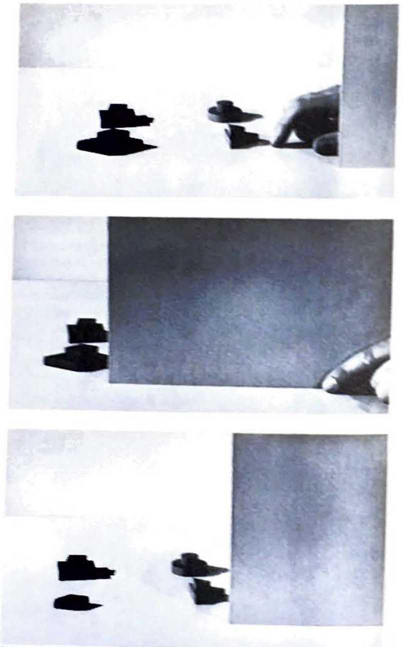
come in pairs, such as shoes and mittens. It may be fun to pretend that the pieces in an eight-block subset are an assortment of socks to be worn in various combinations. Once the first pairing is made, children may be quite readily able to complete the task of pairing the other pieces with the same common differences.

Piece or Pieces Missing From a Subset

Problem Card 3 for the *A Blocks* suggests choosing a shape or a color and playing the game of removing or adding a piece. The same problem could be given on a puzzle card.

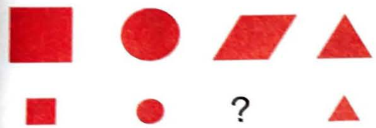


Subset having same shape – random ordering



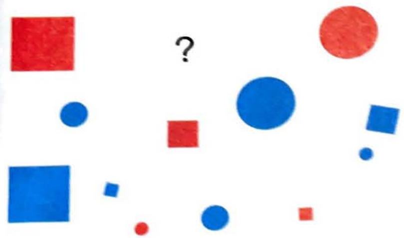
A piece of cardboard is useful as a screen when playing missing-pieces games.

The pieces in the subset can be set out either at random or in some order.



Subset having same color – systematic ordering

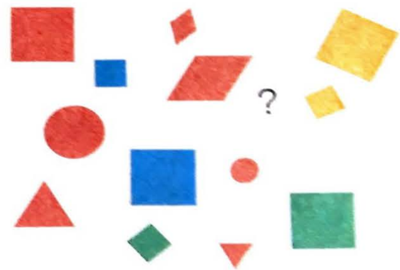
If the pieces are distributed at random, the problem is similar to that involved in generating the set in the first place. It is useful to include cards which have different sizes or different colors from those included in the set of *A Blocks*.



Two values of color, two of shape, three of size

It is possible to play this missing-piece game with different subsets, some of these can be quite challenging. In this case, students may be able to identify the missing piece from the subset without being able to say how the subset was determined. The subset in this card has all but one of the members of the set consisting of the union of red and squares. The rule can be stated: red or square. This gives quite a different subset from the intersection of red and square which produces just the red squares!

Such problems involving unions are perhaps more appropriate for older children, though it would be worth trying some with younger children who have found the previous problems easy.



A Blocks 18

Take all the yellow and all the green circles and diamonds and arrange these eight blocks in some orderly way. One of these pieces is

not yellow
not a diamond
not small

Which piece is it?

Another piece in the set is not large, is a circle, and is not yellow.

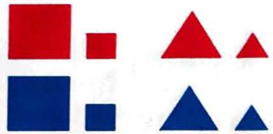
Which piece is this?

Notice that in these examples some values have been stated negatively – it *is not* yellow – while others have been stated positively – it *is* a circle.

Try this game with a partner. Take turns asking about different combinations of positives and negatives.

Can you play the game without looking at the pieces?

Here is the **eight-piece subset**:



It is not red. Since the **piece** you are thinking of is **not red**, all the red blocks are removed and only these remain:

It is not square. Since **your piece is not square, it must be** one of the two triangles.

It is not small, so it **must be the *large blue triangle*.**

COMMENTARY: A BLOCKS 18

The game introduced on this problem card can be **particularly helpful** in the development and use of efficient classification schemes. **It is** similar to one the children may already know, "Twenty Questions," except that with this subset of blocks it is possible to **find out the name** of the piece someone is thinking of by **asking just three questions. Older** students should have no trouble understanding **the problem and should** be able to play the game directly from **the card.**

The first thing in helping younger **children play this "three questions"** game is to provide step-by-step information **that will lead to the selec**tion of a particular piece from a given array. **Assume that you are using** a subset composed of two sizes: *large and small;* two colors: *red and blue;* and two shapes: *triangle and square.*

Until children are familiar with **the game, it is probably best for you to** provide the information all in one form – either **all negative statements** or all positive statements. If you make **all the statements positive, the** game is relatively simple, and for some children **this may be the best** way to start:
I am thinking of a piece: it is small; it is blue; and it is a triangle. Which one is it?
Once the children can identify pieces **from positive information you can** proceed to all-negative statements:
I am thinking of a piece: it is not red; it is not a square; it is not small. Which one is it?
Children may have trouble with negative statements **at first** because **each** statement carries with it an implication: if the piece in a subset **such** as the one above is not red, it must be blue; if it is not square, it **must** be a triangle. If children **have** difficulty with such **implications,** there are two procedures which might be followed: **you could work with** a more restricted subset of only four pieces or you **could use the eight-piece set** but help the children see the effect of each **statement** by actually **removing**, or having them remove, the pieces which **the statement eliminates.**

Removing pieces may help some children, **but they should** be encouraged to try the problems in their heads first – without moving or touching any of the blocks; otherwise their **learning** can become a blind, mechanical process.

Once the children are familiar with this **procedure, begin** to alternate positive and negative information in your statements. Using the same example, the form would be something like this:
I am thinking of a piece that is blue; it is not square; it is not small. Which one is it?

Keep changing the pieces in your eight-piece **subset** and play the game until the children can answer quickly and confidently. Like all the games **on these** cards, this one should be played by children **with each** other, as well as with you. You may want to turn the game **around** and have the children present statements to you, or you might **ask** questions of **them.**

A further challenge for children who **have learned to play** the game with the pieces in front of them is to have them play it with **their** eyes closed, or simply with **information** about the subset but without **the** actual blocks.

The next stage is for children to ask questions about a **subset** in order to determine which piece you, or a child, have **in mind.** Still considering the subset shown above, the questions and answers might go like this:
Is it red?
No.
Is it a triangle?
Yes.
Is it small?
No.
Then it must be the large ***blue triangle.***

Eventually children will become so adept **at dealing** with the eight-piece **subset** that you may want to move on to **consider the** entire thirty-two piece set. You may again wish to start by making **statements** about the piece in question, rather than by having children ask questions.

Playing **these** games with the entire set presents great challenge unless the set is organized in some way. **We** suggest that before making statements, or having children ask questions, about the entire set, the blocks be set out in some sort of *matrix*, as suggested in the next card.

There is an advantage, when children are familiar with the "Twenty Question" format, in starting off *without* having a specific piece in mind. You can allow the values of each attribute to be determined by the child's questions by saing "No" to each of his first three questions about an attribute. With some children the questioning might even proceed as follows:
Is it green?
No.
Is it blue?
No.
Is it red?
No.
Is it a square?
No.
Is it either a triangle or a diamond?
No.
Is it small?
No.
Then it must be a large yellow circle.

Since there are now four values of each attribute, your statements can present information in a number of ways, and the concept of implication can be broadened. If you say, for example, "I'm thinking of a piece that is not red, not green, not yellow," then it must be blue. But if you say, "I'm thinking of a piece that is not red," further questioning is necessary to identify the particular value you have in mind.

A Blocks 19

Set out the following pattern of small blocks.

red square			red diamond
	green triangle		
		yellow circle	
			blue diamond

Can you complete this arrangement using the rest of the small blocks?

Ask someone to remove one of the pieces when you are not looking.

Can you tell which one is missing?

All the blocks in a row have the same color, and all the blocks in a column have the same shape. Such an arrangement is called a *matrix*. Leave the matrix you have made on the table when you are ready to go on. You will use it again for Card 20.

COMMENTARY: A BLOCKS 19 AND 20

These problems present matrices as a method of classification. Some children seem to grasp the organizing power and the elegance of the matrix almost immediately, and are quickly able to solve fairly complicated take-away problems. Many five-year-olds – and many adults – have become fascinated by the way in which the matrix, with its dual organizing principle, makes quite simple what might seem at first glance an exceptionally difficult task.

Some children may need help in analyzing the matrix before they are able to make use of the principles involved. You might say, for example,
Can you point to all the square pieces?
Where are all the reds?
Where are the large pieces?
Some children may find it difficult at first to name a piece which is removed (or covered by a card or a cup). In order to state what is missing they must coordinate two separate ideas – the idea of shape and the idea of color, each associated with one direction in the matrix. They may find it difficult to shift attention from the rows to the columns and back again. These children may be helped if the matrix is placed on a sheet of cardboard or on a turntable and rotated slowly so they see the rows and the columns alternately. Surprisingly, some children actually seem to find complicated problems, involving the removal of several pieces from a double matrix, easier if the matrix is being rotated in this way.

A four-by-four grid made with ¼" tape on a table or a ruled piece of cardboard will encourage young children to try matrix building.

A Blocks 20

Using the large blocks make a matrix which is different from the one you made with the small pieces (Card 19)

Without changing the positions of the pieces within either matrix, place the entire small-block matrix on top of the large-block matrix. That is, the upper-left-hand-corner small piece will go on top of the upper-left-hand-corner large piece, and so forth

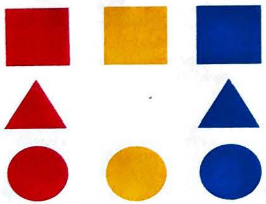
One matrix stacked on another in this way is called a *double matrix*

Ask someone to remove one stack of two blocks from the double matrix

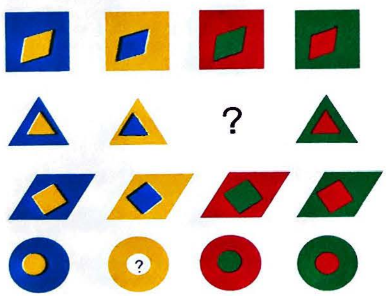
Can you tell which pieces are missing?

Take turns removing stacks and naming the missing pieces until you and your partner can name the missing pieces when at least three stacks are removed at once

The single and double matrix can be used at many different levels. Quite a range of difficulty can be built into matrix puzzle cards.

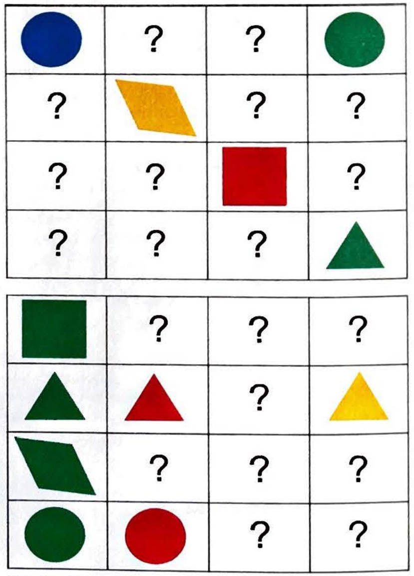


Three-by-three matrix.



Double matrix.

Matrix building can also be stimulated by puzzle cards.



Incomplete matrix puzzle cards.

A Blocks 21

Choose two colors and two shapes different from those you chose for Card 16, and again use both sizes Pair the pieces so that the only difference between the two pieces of *each* pair is color looking at *all* the pairs, color is the *common difference*

How many pairs do you have?

How many ways are there of pairing the pieces so that there is one common difference?

How many ways are there of pairing the pieces so that there are two common differences?

How many ways are there of pairing the pieces so that there are three common differences?

Can you find out the total number of possible pairs for this eight-piece set?

Did you include duplicate pairs?

See if you can figure out some method of keeping track of the pairings you have made

COMMENTARY: A BLOCKS 21

Some students may be able to analyze the problem and answer all the questions on Card 21 without using the blocks. Here is the kind of analysis that is involved:

There are three ways of pairing the pieces so that in each pair there is the same difference – that is, so that there is one common difference.

There are also three ways of pairing the pieces so that there are two common differences.

There is only one way of pairing the pieces so that there will be three common differences.

The problem becomes more manageable when some sort of recording system is used. Here is one form of tabulation:

Number of Differences

Common Difference	One			Two			Three
Color	✓			✓		✓	✓
Shape		✓		✓	✓		✓
Size			✓		✓	✓	✓

Each of these seven different ways of pairing produces four pairs so there are twenty-eight possible pairs.

A different way of analyzing this problem is to reason that every block in the set can be paired with each of the seven others. There will be fifty-six pairs, half of which will be duplicates.

Younger children may enjoy this problem but they will probably approach it less analytically. You may wish to make use of the attribute stickers to keep track of the pairs as they are formed. One procedure which we have found useful is to present a child with a set of eight stickers and a card on which to place one pairing. Before he sticks further pairs, he can be asked to check the cards carefully to see that there are no duplications. It may take several days before all seven different ways of pairing are discovered.

When the seven cards have been completed, children can be encouraged to name the basis for the pairings on each of them. Questions such as "How are the pairs on this card different?" or "Where is the color-difference card?" may help them become more aware of the basis for the pairs they have made.

There are a number of games which can be played with these cards. It is interesting, for example, to make a rapid search for a pair which is named:

Where is the pair of small yellows?

Where are the red triangles?

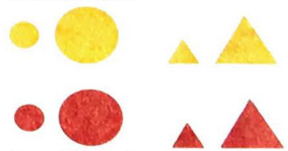
Where are the large circles?

Where is a pair which has only yellow in common?

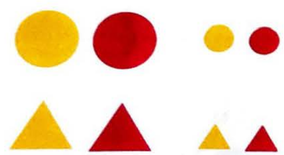
Some Examples of Pairing by Common Differences

This illustration of pairing by common differences provides a good basis for making puzzle cards with the attribute stickers. Any one of these cards could be made into a puzzle card by removing one of the shapes. Of course, it would become quite monotonous to use just these shapes and these colors every time.

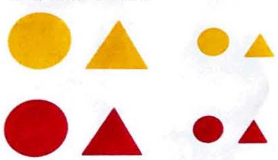
One Common Difference



Common Difference – Size



Common Difference – Color

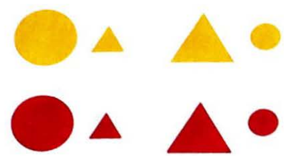


Common Difference – Shape

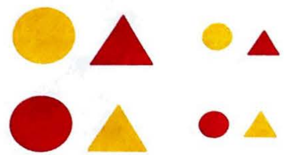
Children who are able to identify a missing piece from a subset containing two shapes, two sizes, and two colors may have trouble when the same problem is presented in puzzle card form. If this is so they can often figure out what is missing if they get the *A Blocks* corresponding to those which are on the card.

It may frequently happen that children find one-difference pairings easier than two- or three-difference problems. A set which would be useful for five-year-olds, therefore, might consist of six or eight one-difference cards, followed by an equal number of two-and three-difference cards. The single difference cards can often be figured out at a glance without analysis of the difference. Although pairing the pieces encourages analysis, children may well figure out which pieces are missing by checking the contents of the entire set, rather than by looking at likenesses and differences in the pairs.

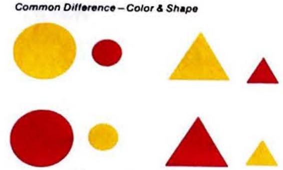
Two Common Differences



Common Difference – Shape & Size

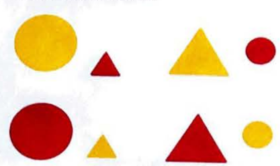


Common Difference – Color & Shape



Common Difference – Color & Size

Three Common Differences



Common Difference – Color, Shape & Size

Children who are having trouble with the cards can be helped by your posing a number of problems for them in rapid succession with the *A Blocks*.

A Blocks 22

Ask someone to wrap one of the blocks in a piece of cloth so that you can feel but not see it

What can you tell about this piece just by feeling it?

Can you tell any more about it by looking at the remaining pieces in the set?

A Blocks 23

Play this game with a partner using all the *A Blocks* Mix the pieces and separate them into two approximately equal groups one for you and one for your partner It is important that you do not see each other's groups so try to separate the pieces by touch without looking at the blocks Put one group into the top of the *A Blocks* box the other into the bottom

Start the game by naming a piece in your partner's collection You should be able to tell what he has in his box by looking into your own box Your partner then takes the piece you have named out of his box and puts it on the table where both of you can see it

Now it is his turn to call for a piece from your box by looking at his own group and at the piece on the table You continue to take turns Each looks into his own box and at the pieces on the table and identifies a piece in the other person's box

COMMENTARY: A BLOCKS 22

This game may be especially appealing to younger children, though it need not be restricted to them. Although only shape and size can be sensed by touch, the child may determine the color of the piece hidden in the cloth by a process of elimination when he has the rest of the set in front of him. Children may be surprised by their ability to solve this problem.

COMMENTARY: A BLOCKS 23

This game is similar to the other naming games suggested. This time, however, each player makes a hypothesis about the pieces in his partner's box by looking at the pieces in his own box and at those already on the table. The strain of making up hypotheses, testing them, perhaps rejecting them, and starting again, may make some children uncomfortable until they are able to work out efficient ways of arranging and scanning their blocks.

The game may be simplified if it is turned into a team endeavor, two or three children working with each box, taking turns making hypotheses. Another way to make the game easier is to put the pieces which are taken from the boxes out on the table in the form of two matrices side by side, one for the large pieces, one for the small pieces; this makes it easier to determine missing pieces and relates this game to the matrix games suggested on Cards 19 and 20.

A Blocks 24

Card 23 presented a "two-box" game. Here is a "one-box" game. Spread all the blocks on the table. Invent a rule that will tell which pieces go into the *A Blocks* box. For example, one rule might be that all the big red pieces go into the box. Ask your partner to pick up any piece, tell him whether or not it belongs in the box according to your rule. If it does, he puts it in the box. If not, he sets it aside. He keeps testing the pieces, one at a time, to see whether or not they belong in the box until he discovers what your rule is. The object of the game is for him to find out your rule after testing as few pieces as possible.

There are many other possible rules. One rule could be that all the red pieces and all the yellow pieces go into the box. Still another rule could involve putting into the box all the green pieces and all the triangles. You could say that all the square *not-blue* pieces (that is, all the pieces that are both square and not-blue) go into the box.

You and your partner should take turns making up rules and figuring them out.

COMMENTARY: A BLOCKS 24

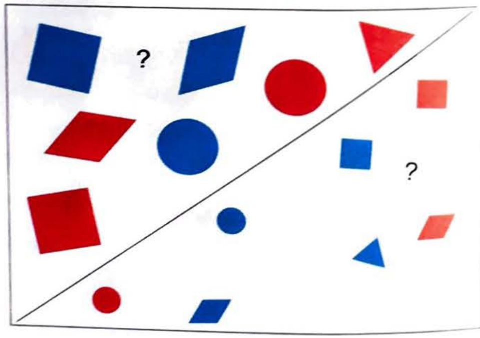
This game requires children to discover a rule by testing specific instances to see whether they conform to it. It provides an opportunity for them to apply the kind of reasoning they have been doing about class relationships and to extend their explorations to problems which are more and more complex. The game can be played at many levels of difficulty. A simple rule to start with might be that all the big pieces could go in the box. An advanced rule might be that the box could contain all the not-green, not-circles – that is, all the pieces which are not green *and* all the pieces which are not circles. This game is suitable for a wide range of students because of the many possibilities for making rules.

In making up rules students may shift between those which involve intersections (for example, *small green pieces*) and those which involve unions (for example, all the *small* pieces *and* all the *triangles*). Although neither of these terms has been introduced on the cards, students who have worked through the problems on Cards 8, 9, and 10 have been dealing with unions and intersections, and those who have studied sets in their mathematics classes may be aware of the distinction. It is not necessary to know the terminology, however, in order to play the games satisfactorily.

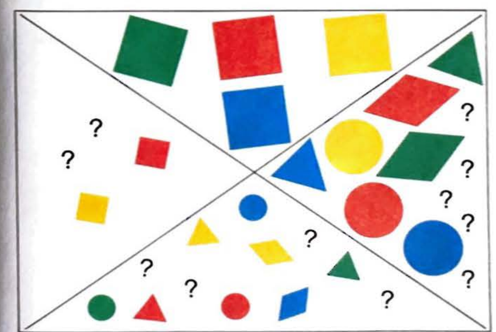
The game requires considerable flexibility in forming and rejecting hypotheses about the rule if one is trying to choose as few test pieces as possible – that is, if one is trying for an elegant solution.

The requirements of this game may encourage students to look for methods of testing more than one piece at a time. For example, if the first few pieces which do go into the box are all either red or yellow, a reasonable hypothesis might be that none of the greens or blues go in, and the student might group together all the greens and the blues and ask whether any of the pieces in the group go into the box. Depending on the answer, further groupings could be tested, or individual pieces could be tried. The game permits the use of a wide variety of strategies.

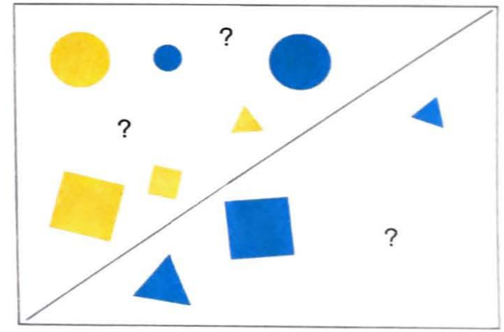
In this case, a subset of red and blue circles, triangles, squares, and diamonds has been used, the large pieces being separated from the small.



In this problem one line separates the squares from the not-squares, and the other separates the large from the small pieces. This is a variation of a two-loop problem.



From a set of yellow and blue squares, circles, and triangles, a subset is formed of pieces which are either circles or yellow.



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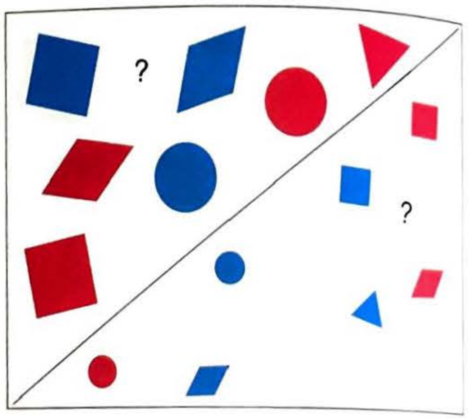

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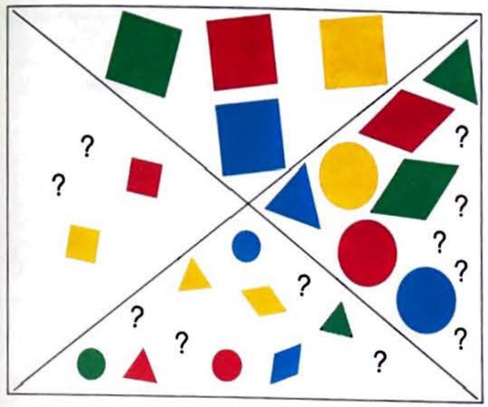

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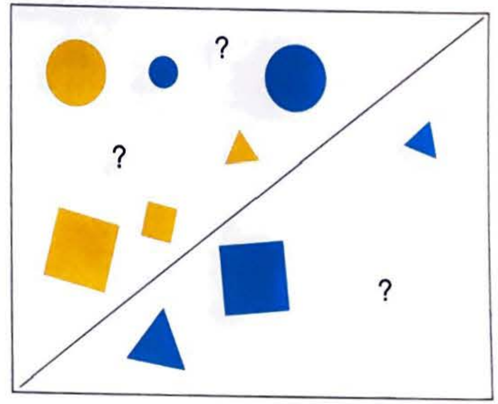
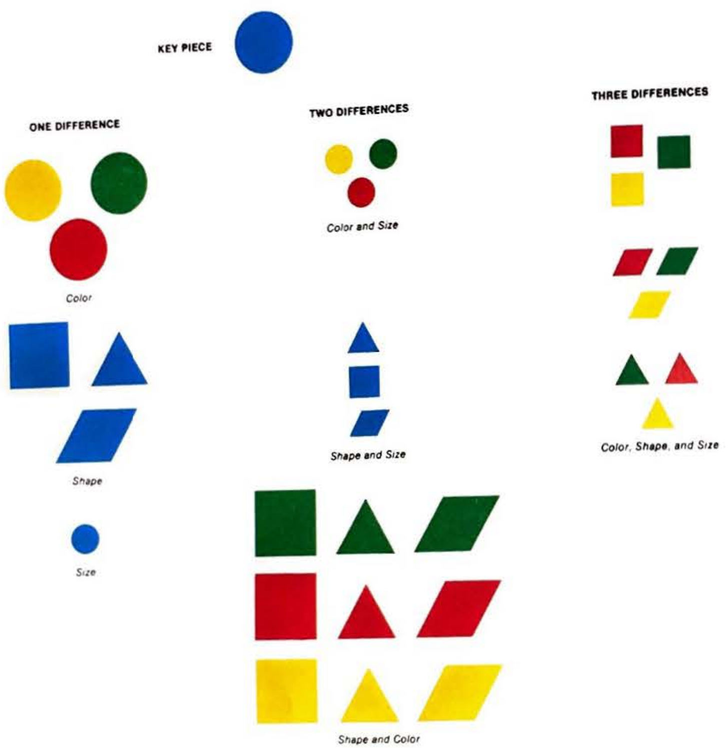




A Blocks 25

Take three of the loops from the *A Blocks* box and put them on the table

If they take up too much space, double them up like this

Choose a block – it does not matter which one. Place this block on the table outside of the three loops. In the first loop place all the blocks that differ from it in one way. In the second loop place all those that differ from your chosen block in two ways. In the third loop, place all the blocks that differ from it in three ways.

Choose a different block and play the game again. Practice until you can do it quickly and easily.

A Blocks 26

Place a block on the table. Find a block that differs from it in only one way and lay it next to the first block. Now find a block that differs from the second block in only one way and lay this on the table in line with the first and second blocks.

Can you arrange all the *A Blocks* in a single line so that all adjacent pieces differ from each other in only one way?

Try to make your line into a circle. This requires finding end pieces that differ in only one way.

Can you lay out a figure eight in which all adjacent pieces differ in one way only?

Notice that the center block – at the crossover – has four adjacent pieces. All four must differ from the center block in only one way.

Can you make circles and figure eights with two differences between adjacent pieces? Three differences?

COMMENTARY: A BLOCKS 25

It is not hard for most people to form subsets of blocks which share a single value of an attribute. "All the reds" or "all the squares" can be singled out quite readily by eye, and one can have a simultaneous awareness of the whole problem. The task of making subsets of pieces which differ from a key piece in one, two, or three values is, however, more challenging: it requires the ability to shift back and forth between simultaneous and sequential thinking.

The subsets having a single difference can be broken into three further subsets: those which differ in shape, those which differ in color, and those which differ in size. The two-difference subsets will contain the pieces which differ in color and shape, color and size, and shape and size, while the subset of pieces which differs in three attributes cannot be broken into further subsets.

The tabulation on page 36, based upon the key piece *large blue circle*, may make the analysis of the problem clearer.

COMMENTARY: A BLOCKS 26

These problems extend the experience gained in classifying by one-, two-, and three-attribute differences to a series of increasingly complex tasks. Indeed, the last task suggested, making a figure eight out of pieces with three differences between them is impossible if you use the full set of *A Blocks*. The impossibility of completing the three-difference figure eight may be discovered through trial and error by many children, only some of whom will work out, analytically, the reason for the impossibility.

A Blocks 24

Card 23 presented a "two-box" game. Here is a "one-box" game. Spread all the blocks on the table. Invent a rule that will tell which pieces go into the *A Blocks* box. For example, one rule might be that all the *big red* pieces go into the box. Ask your partner to pick up any piece; tell him whether or not it belongs in the box according to your rule. If it does, he puts it in the box. If not, he sets it aside. He keeps testing the pieces one at a time, to see whether or not they belong in the box until he discovers what your rule is. The object of the game is for him to find out your rule after testing as few pieces as possible.

There are many other possible rules. One rule could be that all the *red* pieces and all the *yellow* pieces go into the box. Still another rule could involve putting into the box all the *green* pieces and all the *triangles*. You could say that all the square *not-blue* pieces (that is, all the pieces that are both square and not-blue) go into the box.

You and your partner should take turns making up rules and figuring them out.

COMMENTARY: A BLOCKS 24

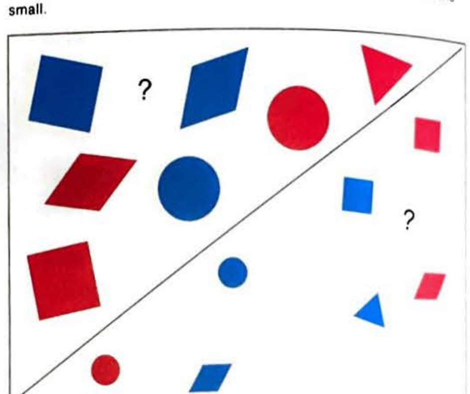
This game requires children to discover a rule by testing specific instances to see whether they conform to it. It provides an opportunity for them to apply the kind of reasoning they have been doing about class relationships and to extend their explorations to problems which are more and more complex. The game can be played at many levels of difficulty. A simple rule to start with might be that all the big pieces could go in the box. An advanced rule might be that the box could contain all the not-green, not-circles – that is, all the pieces which are not green *and* all the pieces which are not circles. This game is suitable for a wide range of students because of the many possibilities for making rules.

In making up rules students may shift between those which involve intersections (for example, *small green pieces*) and those which involve unions (for example, all the *small* pieces *and* all the *triangles*). Although neither of these terms has been introduced on the cards, students who have worked through the problems on Cards 8, 9, and 10 have been dealing with unions and intersections, and those who have studied sets in their mathematics classes may be aware of the distinction. It is not necessary to know the terminology, however, in order to play the games satisfactorily.

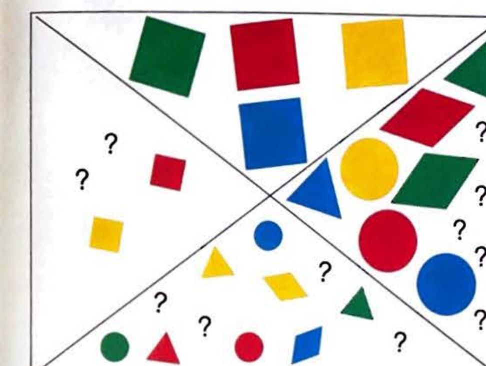
The game requires considerable flexibility in forming and rejecting hypotheses about the rule if one is trying to choose as few test pieces as possible – that is, if one is trying for an elegant solution.

The requirements of this game may encourage students to look for methods of testing more than one piece at a time. For example, if the first few pieces which do go into the box are all either red or yellow, a reasonable hypothesis might be that none of the greens or blues go in, and the student might group together all the greens and the blues and ask whether any of the pieces in the group go into the box. Depending on the answer, further groupings could be tested, or individual pieces could be tried. The game permits the use of a wide variety of strategies.

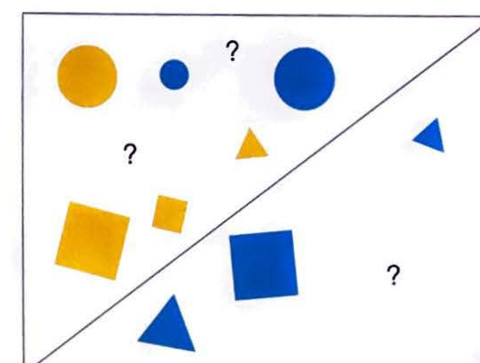
In this case, a subset of red and blue circles, triangles, squares, and diamonds has been used, the large pieces being separated from the small.



In this problem one line separates the squares from the not-squares, and the other separates the large from the small pieces. This is a variation of a two-loop problem.



From a set of yellow and blue squares, circles, and triangles, a subset is formed of pieces which are either circles or yellow.



A Blocks 27

Arrange the *A Blocks* into the following four subsets

blue triangles
not-blue triangles
not-triangle blues
not-blue, not-triangles

Note that the name of each of these subsets refers to values of color and shape. When the names of these values are preceded by *not*, the subset includes all the values *except* the one named. Thus *not-blue triangles* is a subset which contains all the triangles which are not blue. *Not-blue, not-triangles* is a subset which contains all the pieces which are neither blue nor triangles.

Choose another value of color and another value of shape. Make four subsets similar to those listed above. Name them, again using just the names of the values and, when necessary, the word *not*.

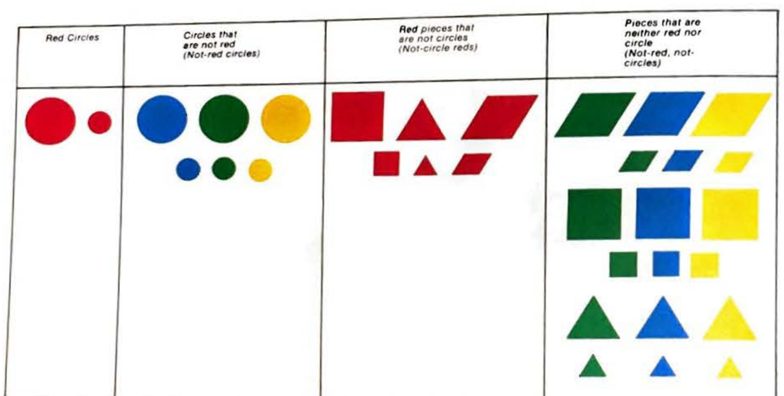
Make subsets with other values such as *yellow* and *circle*, *small* and *not-blue*, and so forth. Practice naming the subsets.

COMMENTARY: A BLOCKS 27

Here the problem is to form subsets named by combinations of values and negations. At first, it may be easier for younger children to select groups as you name them, rather than to name groups after they have been selected. You might start, therefore, by having the children make subsets as you name them, repeating the name after the subset has been formed.

Begin by asking for the subset of blue triangles. Then ask the children to make the subset of not-blue triangles. If this new term is not understood, simply ask for all the triangles that are not blue. Continue naming subsets by values and negations of values.

When children can form subsets easily, present them with already formed groups, such as the four shown below, and ask them for the names.



A Blocks 28

This problem will require two colored loops and the label cards which are included in the box of *A Blocks*. You will not need the labels "N" for this game.

Put any two loops on the table. Put all the red pieces inside one loop and all the square pieces inside the other. Find the label cards for these subsets. All the pieces in the loop must have the value shown by the label card.

What have you done about the red square pieces?

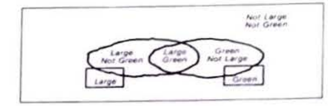
Check to be sure that you have the pieces in the right place. *All* the red pieces and *only* the red pieces must be in the first loop. In the second loop must be *all* the square pieces and only the square pieces.

Make up a different problem. Choose values for different combinations of attributes—color and shape, color and size, or shape and size. Practice playing this game until you can put the blocks where they belong quickly and easily.

Turn the label for each loop face down. See if your partner can identify the labels from the pieces in the loops.

A Blocks 29

Put two loops on the table as you did for the previous proble[m] loop *Green* and the other loop *Large*. If the loop labeled *Green* o[...] *(intersects)* the loop labeled *Large*, you can name the subsets tha[t] in the three spaces created by the intersecting loops, as well as the space outside the loops this way



Suppose you have two such intersecting loops, one labeled *Yellow* and the other *Triangle*.

What are the names of all the spaces, including the spaces outside the loops?

Make up other problems like these.

Can you name the spaces without putting the blocks in them?

Imagine the blocks and loops and try naming the subsets.

COMMENTARY: A BLOCKS 28 AND 29

Card 28 presents in concrete form, using the loops, the same kind of problem dealt with in Card 27. By now, children may be familiar with the characteristics of individual blocks, and they may be familiar with *classes* of blocks, but they may still not be able to shift attention easily from individual blocks (as, the large green triangle) to classes (as all large greens) and back again. The loops enable children to focus on classes without losing their awareness of the attributes of specific pieces. Even though they have been making use of the fact that each block has three attributes, some children may be surprised when the class membership of individual blocks is made visually explicit.

Card 28 does not suggest to students that they overlap the loops; we feel they should have a chance to discover this for themselves. If they do not discover it, Card 29 spells out the problem more explicitly.

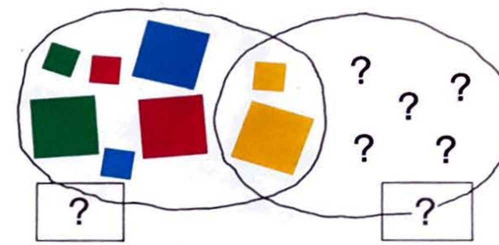
Notice that the space *outside* both loops is as much a labeled space as are the spaces *inside* the loops.

Many first graders have enjoyed two-loop games with the blocks; some have been able to handle three-loop games; and a few have played loop games with negations.

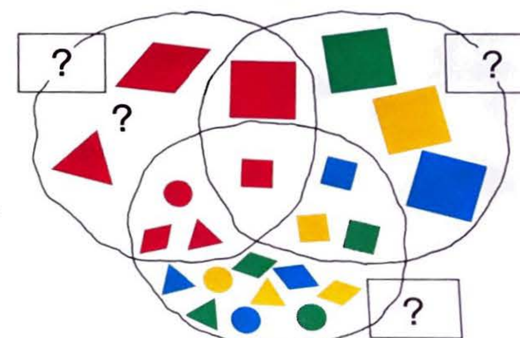
While it is likely that children will find it more difficult to work from a graphic representation than from the blocks themselves, it may be that some young children who have become quite familiar with actual loop games will be able to deal with them on puzzle cards.

The problem can be to determine the labels of one or more of the loops, the contents of the loops, or both. In most instances children will find it necessary to have certain information in order to complete the problem, even when this information is not asked for in the problem. It is probably useful to present these problems in many different forms.

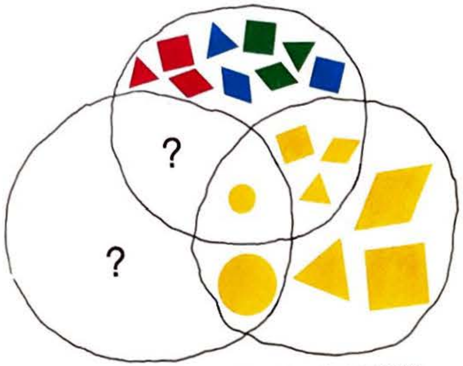
Puzzle cards which call for the use of negative labels are apt to be more challenging. Older students might explore levels of difficulty by making up problems for each other.



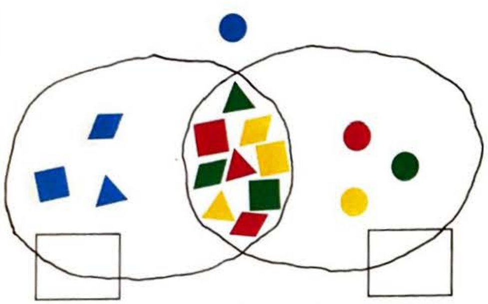
Two-loop problem with unmarked labels and missing pieces.



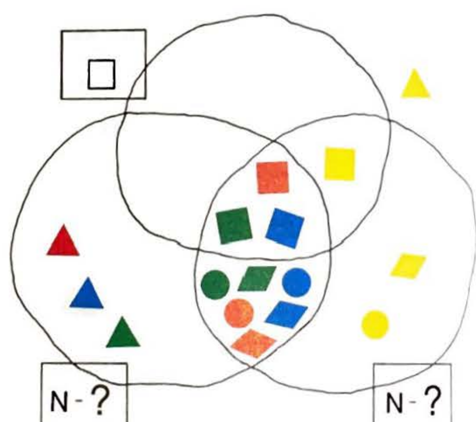
Three-loop problem with unmarked labels and one piece missing.



Some of the contents of one loop are missing. Although the problem is to supply the missing contents, to do so it is necessary to label the loops.



Two-loop problem requiring negative labels. Only one size of piece is used.



Three-loop problem with one positive label given, two negatives required.

A Blocks 30

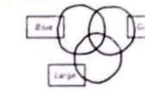
Make the following subsets

small yellow circles
small not-yellow circles
small yellow not-circles
not-small yellow circles
not-small not-yellow circles
small not-yellow not-circles
not-small yellow not-circles

What is the name of the subset of the remaining pieces?

A Blocks 31

Arrange any three loops and label them like this



You will not need the "N" labels.

Place blocks in the spaces where they belong. For example, the space that is in both the *Blue* loop and the *Large* loop but not in the *Circle* loop must have in it *all* the *large blue not-circles*, and *only* the *large blue not-circles*. Check each of the loops in turn to be sure that it has all its pieces and *only its pieces*.

Name each of the subsets you have made, including the subset of pieces outside all the loops.

Make up other problems like this by labeling each loop with different values from those used here.

COMMENTARY: A BLOCKS 30 AND 31

These cards repeat, with three attributes, the type of problem presented with two attributes on Cards 27, 28, and 29. While the tasks are essentially similar to those on the earlier cards, the addition of a third attribute makes the game considerably more challenging. There are eight subsets instead of four. When the three loops are used, as on Card 31, seven of these subsets are inside the loops and one is outside.

As with the two-attribute games, it will probably be easier for children to construct the subsets as they are named than to name already-formed subsets. Many younger children may be able to form these eight subsets if you name the subsets for them. They will need to be quite familiar with the *A Blocks* before they can play the three-loop game described on Card 31.

If, in making up problems of their own, children choose two values of the same attribute to use as labels, they will encounter the *empty set*. We discuss this on Card 34, but the children will naturally discuss it if it arises while they are working on Card 31.

A Blocks 32

Choose one value for each attribute, for example *small*, *blue*, and *square*.

How many subsets can you form using just these values and their *negations*? (*Not-blue* is the negation of *blue*.)

One subset would contain the small blue square, another would contain all the small blue pieces that are not squares, etc.

If necessary, keep track of the subsets you make by listing them on a separate piece of paper.

	Small	*Blue*	*Square*
Subset 1	✓	✓	✓
Subset 2	✓	✓	x
...	...	...	...
...	...	...	...

When you have formed all possible subsets, set out three loops as you did for Card 31, label them with the values you have chosen, and then place each subset in its appropriate space.

Choose a different value for each attribute.

Can you imagine and name the subsets which could be formed using these values *without using the blocks*?

A Blocks 33

Lay out the three-loop pattern using any three loops.

Label each loop with the *negation* (N) of a value. One label might be *N-Green*, another *N-Circle*, a third *N-Small*. Place blocks in the proper spaces. Remember, only the pieces that are not green must go in the *N-Green* loop, and all of them must go in this loop. All pieces that are not circles and only these pieces must go in the *N-Circle* loop, etc.

COMMENTARY: A BLOCKS 32

Cards 30, 31, and 32 present a sequence of activities, each of which builds upon earlier experiences. The games suggested on Cards 33, 34, and 35 may be difficult unless the student is thoroughly familiar with the activities presented on Cards 30, 31, and 32.

On Card 30, subsets were named and the student was asked to make them.

On Card 31, a visual representation of subsets was created by the student, first using values suggested to him, and then using values he decided upon.

Card 32 asks the student to make subsets and then name them. Naming the subsets formed from the combinations of values and their negations calls for considerable flexibility in coordinating a number of ideas. The table suggested on Card 32 may facilitate this coordination. When completed it looks like this:

	Large	Red	Diamond
Subset 1	✓	✓	✓
2	✓	✓	X
3	✓	X	X
4	X	X	X
5	X	X	✓
6	X	✓	✓
7	✓	X	✓
8	X	✓	X

✓ = *positive value* X = *negative value*

When the subsets are formed, placing them in the appropriate spaces in a labeled three-loop pattern may help the student grasp more firmly the structure of the problem. After they have had practice naming the subsets formed when they use the loops, older students may be able to master the complexity of creating eight subsets *without* using the loops.

COMMENTARY: A BLOCKS 33 AND 34

These cards, with Card 35, provide progressively more difficult versions of the three-loop game introduced on Card 31. All of these extensions are difficult, and they will probably prove of interest mainly to older children and adults who have had considerable experience with *A Blocks*.

Card 33 provides systematic practice in dealing with negations. Most people find negative information more difficult to deal with than positive information. This may be, in part, because daily life provides us with less experience in using negations. Also, negations generally leave more possibilities open than do positives. If you consider the *not-blue* pieces, for example, you are dealing with the subset of yellow, red, and green pieces. Only in the case of a two-value attribute, such as size, does the negation eliminate as many pieces as does the positive; *not small* is, of course, large.

Some people find it easiest to handle negations by turning them into positives—by considering the values which are not eliminated. Others may be able to deal with "untranslated" negations from the first. Many people will find that after they have turned negations into positives for a while they will no longer need to do so.

If students make up other three-loop games with negations, in addition to using the combination of values suggested, they may encounter the empty set. If two of the labels are *not-large* and *not-small*, then the intersection of the loops with these labels cannot contain any pieces, since there are no pieces that are neither small nor large. Empty sets are explained to the student on Card 34, where they are most likely to occur.

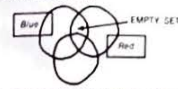
Note: Many children have enjoyed playing a *not game* in a more general context. For example, you might hold up a pencil and ask "What is this *not*?" After collecting *nots* for a few objects which the children may suggest, try asking the question, "What are you *not*?" Children have had fun, after they have listed some of the things they were and were not, drawing pictures of items from their own personal lists. It often seems to surprise children that there are more things that they are not than things that they are. We have found that some children, after listing nouns that they were not ("I'm not an elephant," "I'm not a pencil," etc.), become more adventurous and begin thinking of adjectives ("I'm not hungry"), participles ("I'm not running"), and pronouns ("I'm not you").

Most people find it more difficult to deal with combinations of positive and negative labels than with negations only. Card 34, which suggests that labels be assigned randomly, makes it probable that such combinations will occur. The questions asked on the card are difficult: it is especially difficult to predict the number of pieces left outside the loops by a particular combination of values, and it becomes harder and harder to keep in mind information already gathered about pieces that have been excluded from various intersections. It may help, at some point, to use paper and pencil to keep track of this information.

It would be unrealistic to expect all older students, or even adults, to be ready to respond to the challenge offered by these problems. We have referred to "manageable complexity." It is at this point that these games, while still manageable, become more and more complex, and people differ in their taste for complexity.

A Blocks 34

Once again lay out the three-loop pattern. Shuffle the set of label cards, both positive and negative. Without looking, select three cards. Several interesting combinations of values and negations may occur. For example, two of your labels might be values of the same attribute, say *red* and *blue*. The space made by the *intersection* of the two loops cannot be filled since no block is both red *and* blue. A set (or subset) with nothing in it is called an *empty set*.



Another possibility is that one of your label cards will be for a value and another for its negation.

What happens when one loop is labeled *Circle* and another *N-Circle*?

Which combination of labels leaves the fewest pieces outside the three loops?

Do any combinations leave no pieces outside? Just one piece?

Which combination leaves most of the pieces outside?

Look at the label cards you selected. Before placing the pieces called for in the loops, see if you can predict how many pieces will be left outside.

A Blocks 35

When you feel sure you have mastered the previous three-loop games, find a partner who is also an expert. Start by using only the positive label cards. Choose a label for each loop and place it *face down* by its loop.

Your partner must discover what the labels of the three loops are by placing pieces in the spaces and finding out from you whether they belong there or not. Each of the thirty-two blocks belongs in *only one particular space*, although not all will necessarily go into the loops, and there may be more pieces in some spaces than in others.

In order to give correct answers to your partner, you may want to draw a small diagram for yourself showing the labels you have given to the loops, and refer to the diagram as you answer. You will tell him that a piece is correctly placed only if it is in its unique space—that is either inside one of the seven spaces created by the loops or in the space outside the loops.

A much more difficult version of this game can be created by using only negative labels, and telling your partner that you are using only negative labels. The most difficult version requires you to use any combination of labels, not telling your partner whether they are all positive, all negative, or a combination.

Much of the difficulty of these games stems from the difficulty of remembering what you have learned from each trial of a piece in a space. You may want to figure out some way of recording this information.

COMMENTARY: A BLOCKS 35

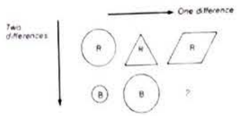
The games suggested on Card 35, all of them difficult, nevertheless range from one which many older children will be able to play quite successfully—the two-player game with only positive label cards—to a game which few adults may be able to manage—the two-player game with labels chosen randomly, leaving open the question of whether they are all positive, all negative, or a combination. It is hard enough to play this game knowing that the labels are either all positive or all negative. When there is the possibility that they are mixed it becomes exceptionally difficult to remember all the information you have obtained from each trial.

A number of first- and second-graders have learned to play the two-loop hidden-label game. The child in charge of the game needs to be able to see the labels, however. Hiding them behind a screen or box has worked very well.

A Blocks 36

In working through Cards 19 and 20 you constructed a *matrix*. In working on Card 26 you made rows, circles, and figure eights out of the blocks, putting together pieces with one, two, and three differences. The problems that follow combine the *matrix* and the *common difference* ideas.

First, arrange all thirty-two *A Blocks* into a matrix so that in the rows adjacent pieces differ from one another in one attribute and in the columns adjacent pieces differ in two attributes. You may have four rows and eight columns, eight rows and four columns, or perhaps even two rows and sixteen columns. The upper left-hand part of a completed matrix might look like this.



Now try a much more ambitious matrix which is something like the double matrix you built earlier, for Card 20. This matrix will consist of sixteen stacks of two pieces each, arranged in four rows and four columns. Adjacent pieces in the rows will differ in one attribute, adjacent pieces in the columns will differ in two attributes, the two pieces in each stack will differ in all three attributes. You will end up with a matrix on top and one on the bottom. Remember that the rows and columns in *both* matrices must observe the one- and two-attribute difference rule. Good luck!

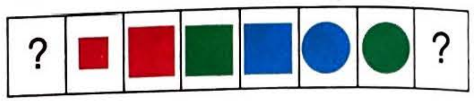
COMMENTARY: A BLOCKS 36

A number of earlier cards (21, 25, 26) focused students' attention on differences. The problems on this card provide additional practice in dealing with one, two, and three differences, and present a formidable challenge for the student. The second problem, involving a double matrix, has taken an adult as much as two hours to solve, but was solved by one thirteen-year-old in fifteen minutes!

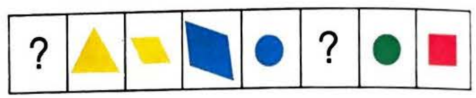
At first, many students will find that they need to focus consciously on the number of differences in rows and columns, and perhaps even form subsets of pieces which can be placed in a given position. One can be very clear about the distinction between the attributes, have had considerable practice in dealing with their values, and still be slow in identifying pieces which possess specific combinations of values, as required for various positions in the matrix. With practice, however, it may become possible to proceed much more rapidly, shifting quickly between simultaneous awareness of the emerging pattern and sequential attention to individual pieces.

It may be advantageous for two people to work together on these problems, especially if difficulty is encountered. It is easy to make mistakes when trying to coordinate differences in attributes in two directions, let alone three, and if two students work together, perhaps taking turns adding to the matrix, they can check on one another.

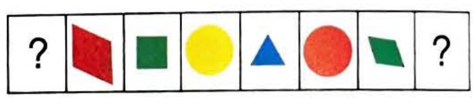
A variety of puzzle cards can also be made which involve differences between adjacent pieces. These could be in a line, a cross, or some other configuration. In most instances there will be several possible answers.



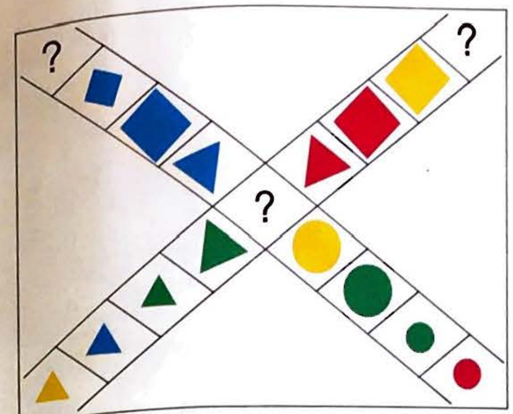
One-difference line



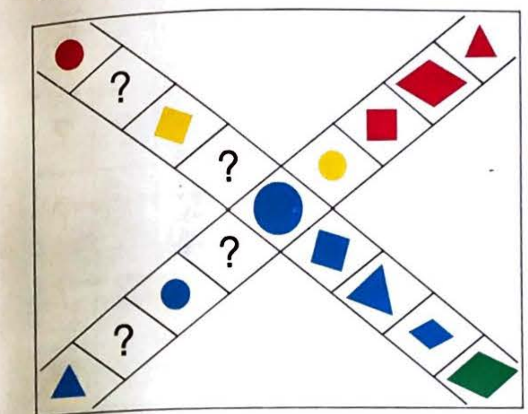
Two-difference line



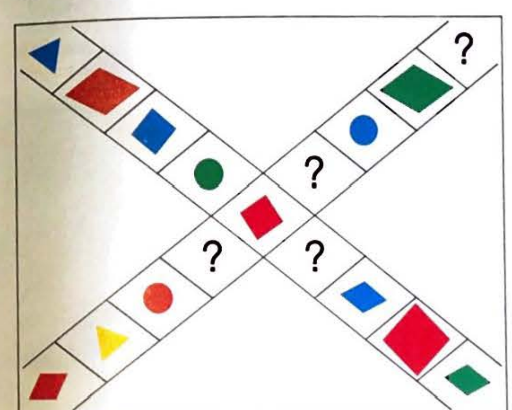
Three-difference line



One-difference cross.



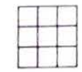
Two-difference cross.



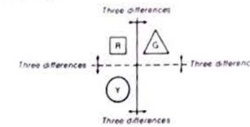
Three-difference cross.

A Blocks 37

Here is a game that is played on a four by four board. You can make one by drawing lines on paper or by putting tape on a table.



The first player places a piece in any of the spaces. The second player may play in any space, but if he places a piece next to one which is already on the board, it must be different from that piece in three ways. If a piece is placed next to two other pieces it must differ from each of the others in three ways.



Can you fill in all the spaces?

When you are able to play this game easily, try changing the rule so that pieces next to each other are different from each other in one way or in two ways.

COMMENTARY: A BLOCKS 37

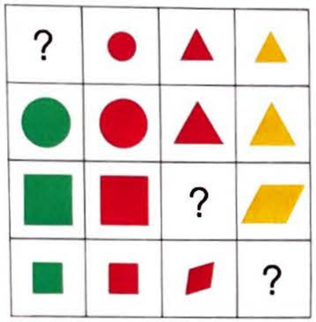
The three-difference game on the four-by-four grid is easier for most players than the two-difference game. The basic problem is the same as that suggested on Cards 25 and 26, but there is the added complication of arranging the blocks so that a piece added is different in the prescribed number of ways from the piece to the left, to the right, above and below.

The point of the game is to see how many pieces can be placed on the board. It may be that more pieces can be placed on the board if some of the other pieces are rearranged. A different game may be played by dividing the pieces at random so that each player is limited to his own group of pieces instead of drawing from a common pool of the complete set of *A Blocks*.

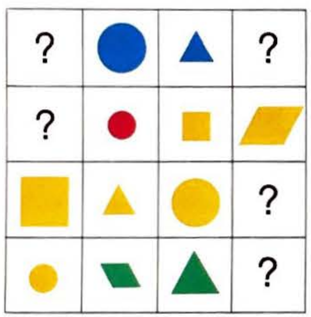
A further variation of the game results if the object is to block the other players so that they cannot move. Students should be encouraged to vary the rules so that they may see how different limitations change the problem.

Another form in which to present this problem is to use either all the large or all the small pieces and to arrange them in such a way that there are two differences between adjacent pieces. All sixteen pieces can be placed on the four-by-four board in such a way that there are two differences between adjacent pairs, horizontally and vertically, though most people will find this quite hard to do. This same problem is stated in a different way on Card 39. Students who have been able to solve the problem in one form but not in the other will be surprised when they see the relationship between the two.

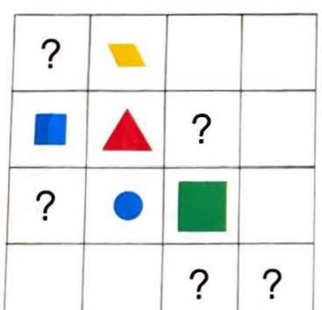
The two-difference and one-difference problems are apt to be quite difficult for most people. It is usually much easier to work on these with the actual *A Blocks*, a four-by-four grid, and a partner.



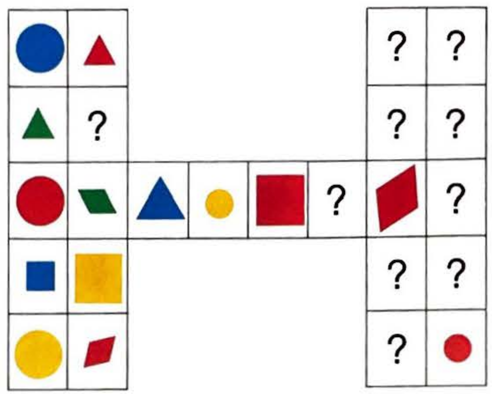
Adjacent pieces differ in one way, horizontally and vertically



Adjacent pieces differ in two ways, horizontally and vertically



Adjacent pieces differ in three ways, horizontally and vertically

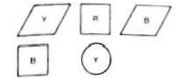


Three-difference "H"

A Blocks 38

This game is played with either all the large blocks or all the small blocks. Divide the blocks at random so that each player has eight pieces.

The first player may put out any piece from his subset. The second player must match it with a piece which is different from it in two ways. Players alternate playing the first piece in each pair.

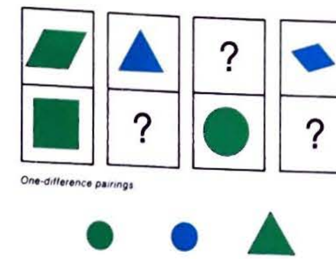


If the first player puts out the yellow diamond, the second player might play the blue square. The second player might start the next pair with the yellow circle, and so on.

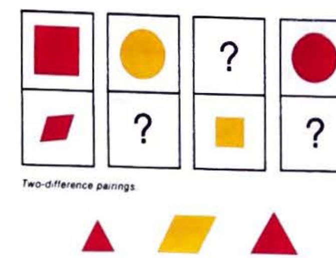
Try playing the game so that all the pieces are used. Another way to play this game is to see if you can force your partner into a position in which he cannot match your piece with one of his own which is different in two ways.

COMMENTARY: A BLOCKS 38

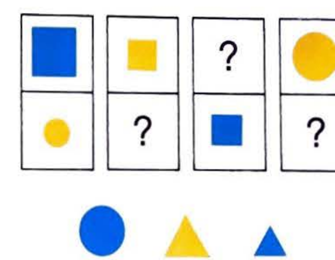
This game can be played on a simple or on an advanced level. The simple game is to match one's partner's pieces by either two-difference pieces or one-difference pieces. Since it is often difficult for children to remember whose turn it is, it might be just as well to have one person do all the placing and the other person all the pairing for a single round. People seem to solve this problem in two different ways. They either check out the pieces one at a time to see if each has one difference or two differences from the piece already played, or they classify by differences so that they identify all the pieces that can be played and all those that can't. If it is a two-difference game, and a red circle is played, for example, no piece which is either red or circle can be played. Students will find that most of the time they will be able to complete all the pairings, but there will be other times when no further move can be made. If the game is played competitively, it becomes quite difficult to figure out how one can block one's opponent from playing.



One-difference pairings



Two-difference pairings



Three-difference pairings

A Blocks 39

Latin squares are introduced on *Color Cubes* Cards 13, 14, 15, and 16. Latin squares can also be made with the *A Blocks*, though this is more difficult.

Use either the large pieces or the small pieces.

Arrange them in a four-by-four square so that there is one piece of each color and one piece of each shape in each row and in each column.

Don't be surprised if you have difficulty with this problem. Most people find it quite hard. It may help you to try Latin squares with the *Color Cubes* first.

If you are able to do this problem, try to remember how you reached your solution. If you used some system, try to remember what it was and write it down. Study what you have done and, if possible, compare your solution with your neighbor's.

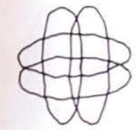
COMMENTARY: A BLOCKS 39

This problem requires one to check both rows and columns for color and for shape. Many people find this difficult. While it is not hard to establish an arrangement that meets either the color requirements or the shape requirements, it is hard to fulfill both requirements at the same time. Transposition of pieces will affect both the shape and the color distribution, of course.

It may be helpful to call the student's attention to the number of colors in the diagonals or in each of the quadrants.

Color Cubes 1

Lay out the red, green, yellow, and blue loops in the following pattern.



Imagine this is a plan for a city. In this city, you use the colored loops to tell what color buildings may be put in the various spaces. You can make buildings of one or more cubes.

Invent a rule or rules which will tell what color buildings go in the spaces. Many kinds of rules are possible. Try some rules which tell you what buildings *cannot* go in certain spaces.

Color Cubes 2

Here is an interesting rule for the four-loop city suggested on Card 1. A cube belongs in every space which is inside a loop of the same color; a cube cannot be put into a space unless it is inside a loop of its own color. Red cubes go in every sub-division inside the red loop, blue cubes in all the separate spaces inside the blue loop, and so forth. Be sure that you have constructed as many buildings as possible.

How many three-story apartments are there? Are there any four-story apartments?

If your partner moves some of your buildings when you are not looking, are you able to find the ones which have been shifted?

Remove the loops carefully without disturbing the buildings.

Can you still tell how the city was planned?

Suppose that you turn the rule around and say that a space may have buildings of every color *except* the color of the loop or loops enclosing it. (You will need two sets of cubes to build this.)

What will the city look like?

COMMENTARY: COLOR CUBES 1 AND 2

The problems suggested on these two cards are suitable for a very wide range in age. Here is a scenario which we have used quite successfully in introducing one form of the game to five-year-olds:

Teacher:

Do you know what a city planner is?

Child:

No.

T: *Do you know what a dogcatcher does?*

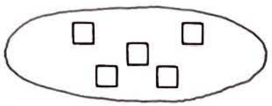
C: *He catches dogs.*

T: *What does a back-scratcher do?*

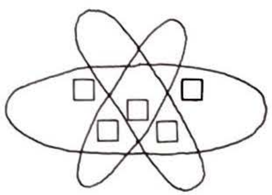
C: *Scratches backs.*

T: *Well, city planners plan cities. I am going to show you the plan for a city and perhaps you can figure out what the city will look like.* (Put out the four loops in the pattern suggested on Card 1.) *This is the plan the city planner has made for his city. These buildings* (color cubes) *can go in the spaces made by the loops, but only certain buildings can go in each space. This red building could go here, for example.* (Place a red cube in one of the spaces which is enclosed only by the red loop and no other.) *Can you find where the rest of the buildings belong? We can take turns trying the buildings in different spaces and I will tell you whether what you have done fits the city planner's rule.*

Children will usually fill in the other single building spaces first, making use of the information which you have given them in placing the first piece. There is no way for them to know in advance what the rule is going to be, of course, so that there is bound to be quite a bit of trial and error. You should make it clear that this is really a guessing game—that the city planner might have made up quite a different rule for this same pattern of loops. It is also helpful to tell children, when they have placed a building correctly, that the space is completely filled or that there is room for more buildings or for making a one-story building into an apartment house.



Cubes enclosed in a ring



The same arrangement with two more rings added

Yes, that is the only building that can go there. Yes, that building belongs there, but it is not the only color that can go in that space. Here you have a three-story apartment house. Is there another space where you could have a three-story apartment house?

Most groups of people, adults as well as children, can complete the "city" when the problem is presented in this form. They may be able to tell which pieces go in each space without being able to state a rule. There is no necessity to put a rule into words, though it may be interesting to see if children want to attempt it. This problem provides a good example of what we have been referring to as "manageable complexity." Students are dealing with a great deal of information simultaneously, but the representation which grows up in front of them enables them to keep track of the tests which they have made and to begin to analyze what is involved. It would, of course, be possible to start with a simpler problem, but we feel it would be a mistake to do so unless children were having a great deal of difficulty. Later it might be helpful to try a two-loop or a three-loop city plan, using rules suggested on Card 2 or ones which the children make up. In making up their own games, children may want to use all six colors of loops and blocks.

One hypothesis which many people seem to explore, in trying to determine the rule, is that it involves what might be called "boundary-matching." That is, they become convinced that a cube can go in a space if *part* of the boundary is a loop of its color, whether or not the space is entirely enclosed by a loop of that color. The idea of enclosure is obvious when there is only one loop. When there are several loops people are apt to focus upon the immediate boundaries of each space rather than the large enclosure. They should understand, if the point arises, that boundary-matching or enclosures are equally good rules for building a cube city.

One of the reasons that these problems provide a challenge may be that they require shifting back and forth from the parts to the whole. Most people concentrate on filling up one space at a time: very few place all the cubes of a given color in the spaces where they belong and then do the same for another color. A good way to help younger children check what they have done is to get them to look at the problem another way. If you suggest that they take a "helicopter ride" around the different sections of the city (moving a finger over each of the loops helps), they can focus on the contents of any one of the loops. Many children see quite readily that the red loop contains red cubes and that there are no red cubes outside the red loop, and that this same pattern holds for the other loops as well. When they have seen this they will be ready to play the other games suggested: moving some of the buildings and detecting what has been changed, and removing the loops and seeing if they can be replaced in the proper position.

Children should be encouraged to look at the patterns which they have made and to talk about what they see. They may notice that one city plan results in all the big buildings being at the center of the city, while another has all the big buildings away from the center.

Another way of looking at the city is to consider the differences in the buildings in adjacent spaces. How is the building which is one space removed from the center of the city different from the center building? When children understand the changes which occur across boundaries, they may be able to predict what the buildings in successive spaces away from the center or in toward the center will be.

Many variations and combinations of rules are possible. Some rules may determine the location of each block precisely while others permit several placements. Here is a sampling of other rules.

Any building may be placed anywhere in the city except inside the green loop which is the park.

Any number of buildings of any color may be placed in areas which are bordered by either two or four colors. No buildings may be built in the other spaces.

A building may be placed in any area which is bordered by a loop of the same color, except that no building can be placed inside a loop of its own color.

Any number of buildings of any color may be placed in any area, except that no building which is the color of the boundary loop on one side of the city is allowed. (One could choose a north, south, east, or west boundary or could decide it in relation to the position of one person at the table.)

Four-story buildings of any color are allowed in the black loop. No building may be built outside the black loop.

Color Cubes 3

Choose nine cubes of each of four different colors. Arrange them in a six-by-six square so that the colors alternate in both directions. If you have chosen red, green, blue, and yellow as your colors, your square might look like this.

R	G	R	G	R	G
B	Y	B	Y	B	Y
R	G	R	G	R	G
B	Y	B	Y	B	Y
R	G	R	G	R	G
B	Y	B	Y	B	Y

Tell your partner to look away. Remove one cube from the corner.

Can your partner tell what is missing?

Take two or three cubes away.

Can your partner put them back?

Can you think of other games that might be played with this pattern?

COMMENTARY: COLOR CUBES 3

The pattern created by alternating cubes of four colors is more complex than a simple two-color checkerboard pattern and yet is structured sufficiently so that most children have little difficulty with one-, two-, and even three-cube removals.

There are several ways to play the game. You can remove a cube, or cubes, and ask the child what is missing. You can remove several cubes, mix them up, hand them back to the child and ask him to replace them in the pattern. You can remove single cubes from various parts of the pattern or a number of adjacent cubes.

One thing which quickly becomes apparent is that it is possible to remove so many cubes that the sense of pattern is destroyed, and the task of replacing or identifying missing cubes becomes either a memory task or an impossible one. Children may enjoy exploring just how much distortion the pattern can withstand before it loses its distinctive properties.

Setting the pattern up on a turntable and rotating it slowly as children attempt to identify missing cubes may actually simplify their task.

This is an excellent game for children to play with each other, and one in which they can easily set problems for you. When children set problems for one another, they often have a tendency to destroy the pattern rather quickly by making wholesale removals and by moving cubes. It may be desirable for you to play this game with them long enough for them to get a sense of what is possible with less drastic removals or rearrangements.

Color Cubes 4

Set up the pattern of cubes as suggested on Card 3
Here is a game which can be played by using "imaginary glue"
Pretend that two cubes which are side by side are glued together
Remove them both. Turn them around and put them back

Can your partner tell which cubes have been reversed and turn them back to place?

Take out a square of four "glued" cubes and turn it around

Can your partner put them back in place?

Can you make up harder "glue" problems?

COMMENTARY: COLOR CUBES 4

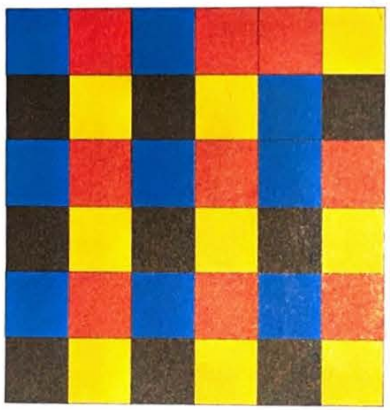
The "glue rule" problems can be extended to provide a challenge for almost anyone. Children may need considerable practice with relatively simple problems before going on to more complex ones. Generally speaking, reversals on the edges of the pattern are easier to detect than those within it. It is important that children be given a chance to devise their own strategies for locating the displaced cubes. Some children, even after a good bit of time, may be unable to find out where the trouble is. You might then suggest that they focus on one quadrant of the pattern, or suggest that they see whether particular rows or columns are in order. It is often surprisingly difficult to determine exactly which cubes are involved in a rotation even after the general area has been found.

Four-cube rotations are especially difficult for many people. It cannot be stressed too much how important it is that children have time to think; often, when several children are playing together, the onlookers make comments such as, "Oh, that's easy, I know what it is," or "Come on, Billy, that's a snap." It may be useful to remind these critics that many problems seem easier from the sidelines than they do from the vantage point of the player, but perhaps the children will learn more tolerance when their turn comes and they find out that things are not as simple as they appear. Having all children close their eyes when the rotation is made is another way of overcoming this problem.

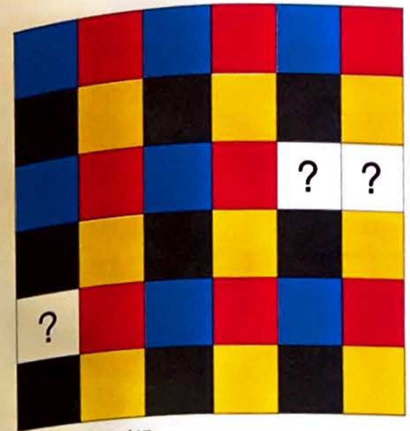
Children may discover spontaneously that it is possible to rotate cubes not only in the horizontal plane but also vertically—that is, to flip them over. If the student has become used to thinking only of horizontal rotations, a three-cube rotation (corner cube and the two adjacent edge cubes) involving flipping over the "L" can be baffling.

A still more complicated version of the game involves double rotations. For example, a block of six "glued" cubes might be rotated, and then four cubes from within the six-cube block might be rotated again, the student's task being to return the cubes to their original position in two moves. (There are at least two ways of doing this.)

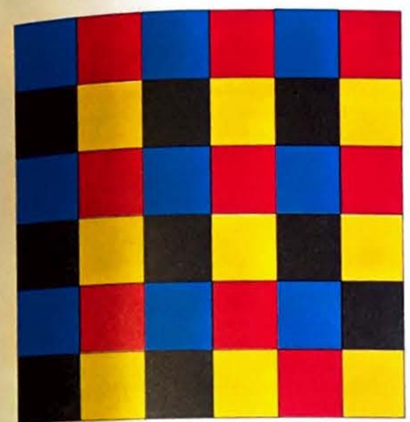
You will find it helpful to use a reusable sticky material, such as Hold-it or another plastic putty, as real glue. This is especially useful for removing and rotating cubes on the inside of the pattern or in groups of four.



Square Color Cubes pattern with one block of four rotated



Three blocks missing from a square Color Cubes pattern

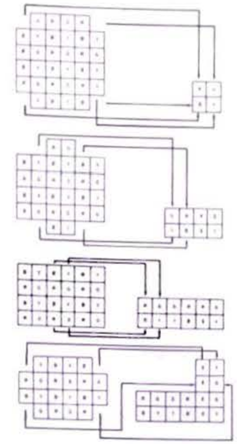


Three pieces have been "glued" together. This pattern can be restored in one move.

Color Cubes 5

Here is the beginning of a transformation according to a systematic rule – one of many which could be used

Try to complete this transformation with the cubes before going on to the card 5A



COMMENTARY: COLOR CUBES 5 AND 5A

This card, in both content and format, is unique among those in these series so far. It does not attempt to present a problem as such; it is designed, rather, to show students a fascinating property of this pattern, namely, that it transforms into other patterns which are related to the original.

After the fifth transformation, the pattern comes back to the original. Some students may be interested in speculating about why this is so; they might want to follow particular cubes through the entire sequence of transformations, perhaps labeling them in some way.

One property of all the patterns emerging from this transformation rule is that they retain a symmetry with respect to the main diagonal, top left to bottom right. That is, if you folded diagonally a sheet of paper which had a diagram or picture of the cubes at each stage you would find that red always folded onto red and yellow onto yellow, while green folded onto blue. Why?

There are many other rules which can be invented for making transformations.

Color Cubes 5-A

If you have completed the transformation pictured on Card 5 correctly, according to our rule, the new pattern will look like this

R	G	G	R	R	G
B	Y	Y	B	B	Y
B	Y	Y	B	B	Y
R	G	G	R	R	G
R	G	G	R	R	G
B	Y	Y	B	B	Y

Using this new pattern as a starting place, apply exactly the same transformation rule to obtain a third pattern. Using the third pattern as a starting place, make a fourth pattern according to the same rule, then a fifth and a sixth

What happens each time?

What happens the final time?

Why?

Can you invent other transformation rules?

Color Cubes 6

Make the six-by-six, four-color cube pattern as you did for Cards 3, 4, and 5

Divide the pattern in half, down the middle

What do you notice?

Divide each of the halves in half. You now have four quarters of the original pattern

What are some of the things you notice about this arrangement?

Push the quarters back together to make the whole pattern

Can you discover other ways of dividing the pattern?

Make the four quarters again. Stack them on top of one. Have your partner look at the tower you've made and see whether he can figure out, from looking all around it, what color cube is under the center one in the top layer

What cube is under that one? On the bottom?

When the four quarters are stacked, there are nine stacks of four cubes each. Are the cubes in any of these in the same order? How many different arrangements of cubes are there in these nine stacks? You might want to separate the stacks into groups of stacks which are alike

Remove the four layers carefully and put them on the table

Can you push them back into the original pattern?

COMMENTARY: COLOR CUBES 6

The activities suggested on this card show the student that there are many ways of looking at the pattern, of breaking it into segments. One method of division which is not suggested on the card is to make nine small squares of four cubes each. Students might also want to explore what happens when the pattern is divided diagonally in various ways—resulting in a steplike diagonal edge.

Students who are quite familiar with the pattern, who have observed closely the four quarters, may find it fairly simple to figure out which cubes lie under the center top one when the segments are stacked.

There is another interesting feature of this particular stack, if the four quadrants have been placed on each other without any of them being rotated. If you look at the four corner piles, you will notice that they are identical: perhaps they are, in order, red, yellow, blue, and green. If you now agree to say that these same colors appearing in inverted form, green, blue, yellow, and red, are really the same order, then among the nine stacks of cubes there are only two different orders of cubes—something which is quite surprising.

Color Cubes 7

Here is a pair of cubes [R] [W]

How many *different* pairs can you make, using six different colors of cubes?

Check to be sure that you have made all possible pairs

Can you find some way of arranging your pairs so that you can be sure that you have found them all?

Can you predict how many pairs you could make using cubes of five different colors? Make these pairs if you are not sure

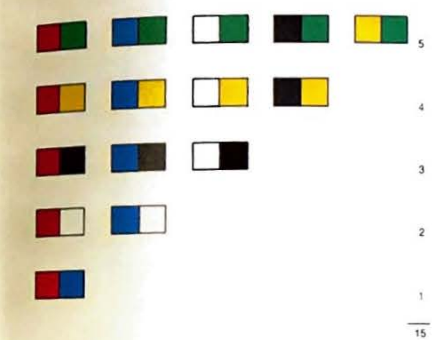
How many pairs could you make, using four different colors? Three different colors?

How many pairs could you make if you had seven different colors of cubes? Eight colors?

COMMENTARY: COLOR CUBES 7

This is a pairing problem similar in structure to that suggested on *A Blocks* Card 17. There are six colors, including black and white. Each color can be paired with five others (6 x 5), but half of the pairs are duplicates.

In constructing these patterns, students probably will find it necessary to set the cubes out in some orderly fashion to keep track of what has been done. Here is one such systematic arrangement:



There are enough cubes in one set to make all the six-color combinations. Students who try to solve this problem with more than six colors will have to work out another system of representation or will have to generalize what they have already learned from the previous cases.

Color Cubes 8

Suppose you have two cubes of different colors—a red one and a yellow one

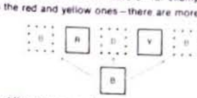
There are two ways of putting them in rows: the red one can go first

[R] [Y]

or the yellow one can go first

[Y] [R]

If you have three cubes of different colors—for example, if you add a blue cube to the red and yellow ones—there are more ways of making a line



How many *different* ways are there of putting three cubes in a row? (A row is different even if, by reversing it, you could make it the same as another row.) Set out all the possible orderings of three cubes of different colors

How many *different* ways are there to put four cubes in rows? (You can make these rows if you use two sets of *Color Cubes*.)

Can you discover a way to predict the number of possible rows for any particular number of cubes without actually using cubes?

COMMENTARY: COLOR CUBES 8

Card 8 contains a hint which may lead children to work out a way of analyzing the problem of ordering (permutations) with any number of cubes. Since there are two ways of ordering two cubes, and in each of these there are three possibilities for placing a third cube, there are six (3 x 2) possible orders for three cubes. If four colors are used, there will be four possibilities for placing the fourth cube in each of the six three-cube arrangements—4 x (3 x 2)

Children should be encouraged to make all twenty-four possible orderings for four colors of cubes. They will need two sets of *Color Cubes* to do this. It will be difficult for them to determine whether or not all the orderings have been made unless they proceed systematically. Those who are successful in developing a systematic approach may be able to analyze what they have done. Students who keep track and make sense of what they have done are frequently excited when they recognize the general principle by which they have been operating.

Color Cubes 9

Choose cubes of three different colors, for example, red, blue, and white

How many different sets or subsets can be formed using three or fewer cubes? (Changing the order of the cubes does not make a different subset.)

One subset would be all three cubes together 

There are three subsets of single cubes

Can you make three subsets of pairs of cubes?

One subset would include no cubes at all! ( )

Counting the subset with no cubes in it and the set which has all three cubes, there are eight sets and subsets possible when you have three cubes

How many ways are there of forming subsets with four or fewer cubes?

Choose four colors and make these subsets

Does this remind you of anything you have done before?

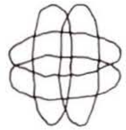
When you have arranged all the subsets of four or fewer cubes, leave these on the table to be used for the next card

Can you calculate (on paper) how many subsets of cubes you could make if you used five colors?

Color Cubes 10

Begin with the subsets of four or fewer cubes that you had at the end of Card 9. Choose four loops to match the colors of the cubes

Place the loops in the four-loop pattern



Suppose that this is another city-planning game, but the buildings are already constructed. Your job is to place them where they belong

Do you have enough buildings to fill the spaces?

Are there some buildings left over?

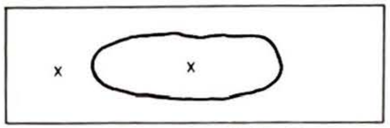
COMMENTARY: COLOR CUBES 9 AND 10

Cards 9 and 10 present problems involving combinations similar to the problem given in the city-planning game on Card 2. The difference is that in the city-planning game there were a number of duplications. Card 10 should help students see the relationship between combinations and city planning.

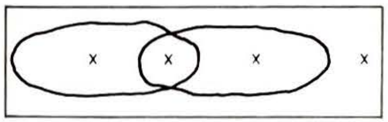
Although the problems presented on these cards have dealt with combinations of values of *A Blocks* and *Color Cubes*, students may begin to see that the same kind of reasoning applies to sets of all kinds. Combinations can be made using any collection of objects.

Mathematical Footnote to Color Cubes

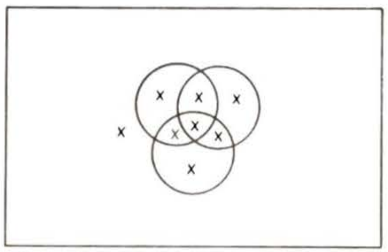
The number of subsets which can be formed from a set with a given number of elements can be expressed as powers of 2, provided empty sets are also counted. Students who have been using the loops will be familiar with the following:



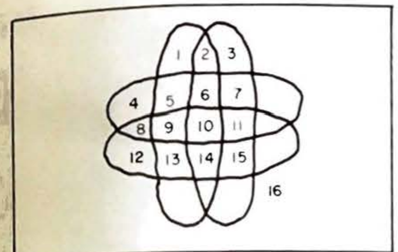
With one loop there are two possibilities. (2^1)



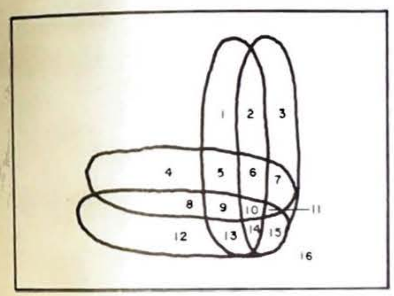
With two loops there are four possibilities. (2^2)



With three loops there are eight possibilities. (2^3)



With four loops there are sixteen possibilities. (2^4)

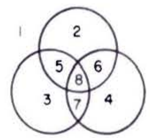


The six duplicate spaces can be eliminated by pulling the loops to one side.

Another way of viewing this is to consider the number of subsets which can be formed from a set containing a certain number of elements. If there are three elements, for example, there will be one subset that has no elements, and also one subset with three elements. There will be three subsets with one element and three more subsets with two elements.

If one chooses three colors of cubes and forms all possible subsets, these subsets can be placed in the three-ring pattern directly:

1 (empty)
2 red
3 blue
4 yellow
5 red-blue
6 red-yellow
7 blue-yellow
8 red-blue-yellow



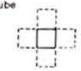
Analyzing the number of possible subsets for a given number of elements can be a challenging but rewarding undertaking. The tabulation of these possibilities is known as *Pascal's Triangle*.

Number of elements in set	Number of subsets having the following number of elements 0	1	2	3	4	5	Total
0	1						$2^0 = 1$
1	1	1					$2^1 = 2$
2	1	2	1				$2^2 = 4$
3	1	3	3	1			$2^3 = 8$
4	1	4	6	4	1		$2^4 = 16$
5	1	5	10	10	5	1	$2^5 = 32$

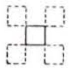
Color Cubes 11

This is a game for two people

Each player chooses one color and takes all the cubes of this color. The first player puts one of his cubes on the table. The second player can play one of his cubes on any side of the first cube

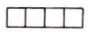


or on any corner, but it must be touching one of the cubes already played



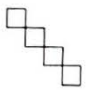
The object of the game is to get four cubes of the same color in a row –

horizontally,



vertically,

or diagonally

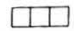


The first person to get four in a row is the winner

Color Cubes 12

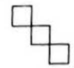
This is a game for three, four, five, or six people
Each person chooses a color and plays in turn
The object of the game is to get three pieces in a row –

horizontally,

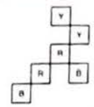


vertically,

or diagonally



It is important that each player place his cube in turn
Here is an unfinished three-in-a-row game



Yellow will win this game unless blocked by both red and black

This game is also interesting when played with the variation of allowing the three cubes to be stacked, one on top of another, in addition to the three other three-in-a-row possibilities

COMMENTARY: COLOR CUBES 11 AND 12

The game of tic-tac-toe, played on a three-by-three board, is an old favorite. The characteristic which makes it interesting, and difficult for children at first, is that a successful strategy requires shifting between offense and defense. Children tend to play either to win or not to lose and they have trouble trying to do both in the same game. Tic-tac-toe gets to be a dull game because neither player can win if no mistakes are made.

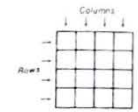
The four-in-a-row game with *Color Cubes* needs no board. It is an interesting variation of tic-tac-toe and provides fresh opportunities for analysis. Children are often more likely to discover winning strategies when they play a number of variations of these games.

Three-in-a-row with two players is a very dull game and has not been suggested on the problem cards. It might be interesting for children to discover this for themselves.

Three-in-a-row with three or more players is an interesting game of strategy. Players will soon realize that they have a choice of which person to block and that it often takes teamwork to keep a player from winning. When there are more than two players, it becomes necessary to think several moves ahead and to make the plays which will be favorable in the long run. When players are familiar with the game, it will be more interesting to have a series of games to determine a winner, since the order of play strongly influences the outcome.

Color Cubes 13

Choose four colors. Take four cubes of each color. Arrange them so that there is only one cube of each color in each row and only one of each color in each column



Any square arrangement in which each row and column has one and only one cube of each color is called a *Latin square*

Try making Latin squares with the *Color Cubes*

Study the color patterns in your Latin squares

Are there different kinds of Latin squares?

Count the number of colors in each quarter of the four-by-four Latin square.

Can you make four-by-four Latin squares which have two colors in each quarter?

Three colors?

Four colors?

What do you notice about the diagonals?

Color Cubes 14

Make Latin squares with three colors of cubes and three cubes of each color

What do you notice about the diagonals?

Move the entire first column so that it becomes the third column

Do you still have a Latin square?

Move the entire first row so that it becomes the third row

Do you still have a Latin square?

What happened to the diagonals in each case?

Try these same things with a four-by-four Latin square

Color Cubes 15

Choose five colors and take five cubes of each color. Make a Latin square with these cubes.

Did you do it by trial and error or have you worked out a system?

Use all six colors and six cubes of each color. Make a Latin square with these cubes.

If you shift rows and columns will you still have a Latin square?

Can you regain your first arrangement by continuing to shift rows and columns?

Color Cubes 16

Make a Latin square using four cubes of each of four colors. Set it aside. Now make a different Latin square using the same four colors that you chose for the first Latin square.

Is it possible to place the second Latin square on top of the first in such a way that no two pairs are repeated? A blue may be on top of a red and a red may be on top of a blue, but there cannot be more than one red on a blue or blue on a red. Only one pair can be made with a cube on another of the same color.

This problem can be done with some kinds of Latin squares but not with others. It will help you to keep a record of the kinds of Latin squares that will fit together in this way.

If you are able to do this problem you can use your solution to solve the problem suggested on *A Blocks* Card 39. Use one of your Latin squares to map the color of the *A Blocks* and the other to map the shape. (For this problem you will be using all the small *A Blocks* or all the large ones.) To map color, you can simply let the colors of the cubes refer to the same colors of the *A Blocks*.

In order to map shape you will need to set up some kind of code. For example, you could let red stand for square, green for triangle, blue for circle, and yellow for diamond. The two separate Latin squares made of *Color Cubes* will determine the Latin square of *A Blocks* if one *Color Cubes* Latin square can be placed on the other one without repeating the pairings.

Even if you are not able to solve this problem go on to the next card.

Color Cubes 17

If you have been unable to solve the problem suggested on Card 16, you may be able to work it backwards. Try the problem suggested on *A Blocks* Card 39, which is to make a Latin square (using either the small or the large *A Blocks*) so that there will be one block of each color in each row and in each column, and also one block of each shape in each row and each column. Then make *Color Cubes* Latin squares, one to map the color and one to map the shape of the *A Blocks* Latin squares.

This problem can be solved. Do not be discouraged if you are unable to do it the first time you attempt it.

If you are able to solve this, study the types of Latin square that you have.

How are they different from other Latin squares which can be made with the *Color Cubes*?

If you have reached a solution to the *A Blocks* Latin square problem, you are ready to try a Latin square with the *People Pieces* as suggested on *People Pieces* Card 16. Save the *Color Cubes* Latin squares that you have used to map the *A Blocks* Latin square. They can also be used to determine a Latin square with the *People Pieces*.

COMMENTARY: A BLOCKS 39
COLOR CUBES 13, 14, 15, 16, 17
PEOPLE PIECES 15, 16

A Latin square is an arrangement of elements (numbers, letters, colors, etc.) such that no two elements are repeated in any row or column of the square. Any number of elements may be used so long as each appears once in each row or column.

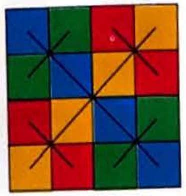
Latin squares with *Color Cubes* present a wide range of complexity. They can be introduced to younger children by using just four cubes, two each of two colors. There is only one way to make a two-by-two Latin square, and it will still be a Latin square if rows and columns are shifted.

A three-by-three Latin square may not be much more difficult for young children to make. There is just one basic type: a solid color in one diagonal and three colors in the other. Rearranging rows and columns will change the color of the diagonal but will not destroy the Latin square.

The four-by-four Latin square is much more interesting. Since solving the Latin square problem with *A Blocks* and with *People Pieces* depends upon choosing certain types of Latin squares, it may be challenging for students to analyze some of the patterns. There are twelve types of four-by-four Latin squares, and a good research problem for older students would be to find ways of telling them apart. In this case the useful attributes are not single qualities, but rather patterns of color or numerical relationships. One way of distinguishing one Latin square from another is to count the number of different colors in each quarter of the square. This is only partly helpful, however, because there are three Latin squares which have two colors in the quarters, four which have three colors in the quarters, and five which have four colors in the quarters. Counting the number of colors in the center block of four gives additional information and, considered with the number of colors in the quarters, helps to distinguish the squares somewhat more precisely.

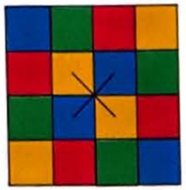
Another useful way to tell four-color Latin squares apart is to draw a diagonal line on the pattern where two or more cubes of the same color touch at their corners. Each of the twelve squares has a distinctive pattern which is revealed in this way.

There are two types of four-by-four squares, as shown on page 62: five of one type, seven of another. The problem suggested on *Color Cubes* Card 16 can be solved if the two squares in the first row or any two of the three squares in the second row are used. Any one of these five squares can be transformed into any other one by rearranging

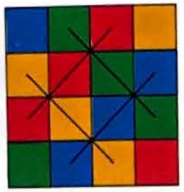


2, 2

4, 4



4, 2



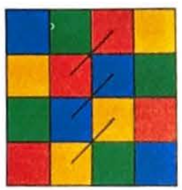
4, 4

2, 4

rows or columns. For example, in the second row, the first pattern can be transformed into the second pattern by moving the first column so that it becomes the third column. Students may be interested in making up a notation so that they can tell how to get from one pattern to another one. (One notation for the move suggested could be. . . .)

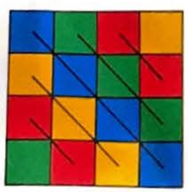
Any one of the second type of square can be transformed into any other of the seven in a similar fashion, but no amount of rearranging of rows and columns will transform a square of the first type into a square of the second type or vice-versa.

3, 4



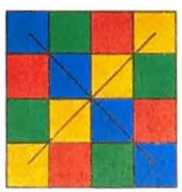
4, 3

4, 4



3, 3

2, 3



3, 2

3, 2

People Pieces 1

Take the lid off the *People Pieces* box but do not look at the pieces Take one of them out and put it in front of you on the table

Can you describe it?

Can you predict what some of the other pieces will look like?

Take out another piece and put it on the table

Can you now make a better prediction about some of the remaining pieces?

Take out the remaining pieces one at a time and put them in front of you Perhaps you can put them in some kind of pattern Each time you take out a piece, try to improve your mental picture of what the remaining pieces may be like When five or six pieces are out, see if you can describe the rest of the set completely Then take out the other pieces and see whether you were right

What can you think of to do with the pieces?

Can you invent some games?

COMMENTARY: PEOPLE PIECES 1

This card presents a task similar to that of generating the set of *A Blocks*, described on Card 1. For younger children the same method of introducing pieces that was described in the Commentary for *A Blocks* Card 1 may be used, except that it is probably a good idea to start by showing the children one of the pieces.

In some ways the *People Pieces* set may be simpler than the *A Blocks* set. While there are four attributes instead of three, there are only two values for each attribute. The main difficulty with *People Pieces* seems to be that of terminology. The words children tend to use spontaneously to describe the pieces refer to more than one attribute: "man," "woman," "boy," and "girl" all refer both to sex and to age. When a child uses the word "big," he may mean fat, he may mean tall, or he may mean both fat and tall.

It is important that children have a chance to play with the pieces, to invent games, and to sort out the various features of the pieces at their own pace, facing the problems of terminology only when these become relevant.

We have used the following terms for the attributes and their values:

Attribute	*Values*
color	red, blue
age	adult, child
sex	male, female
girth (fatness)	fat, thin

Children may be ready to use a standard vocabulary after trying some of the activities suggested on Card 2

People Pieces 2

Put out all the *People Pieces* on the table

What are the *attributes* of this set?

What are the *values* of the attributes?

To help answer these questions, it may be useful to form subsets of pieces such that all the pieces in a subset are alike in only one way

How many pieces are there in a subset that contains pieces which are alike in only one way?

How many subsets are there of pieces which are alike in only one way?

People Pieces 3

Choose two attributes. You might choose color and age, sex and fatness, or any other combination of two attributes. Make subsets which contain pieces having one value of each of the chosen attributes. If you choose color and age, the pieces in each subset will have the same color *and* the same age

How many subsets are there?

Practice making these two-attribute subsets until you can do it easily

Ask your partner to identify the attributes each subset has in common

If your partner makes two-attribute subsets, can you identify the attributes he is using?

How many *different* two-attribute subsets can there be with the *People Pieces*?

COMMENTARY: PEOPLE PIECES 2

While children may have trouble with the terminology for the *People Pieces*, they seldom lack ideas for things to do with them. Many five-year-olds have spontaneously formed the single-attribute subsets suggested on Card 2. Sometimes they have formed subsets of *People Pieces* that have only one difference, that is, three-attribute subsets. Often they invent a story to go with what they have done: "The fat people are all going to eat at this table." "The fathers are taking the boys on a trip." "The boys and girls are playing." Sometimes children group the pieces into families: perhaps the fat red parents have fat red children, or, on another occasion, the fat red parents have thin blue children. (Children often delight in exploring improbable genetic combinations with these pieces!)

After playing with various subsets, children may be able to form others "by eye" before they are able to say in words what the attributes and values are.

COMMENTARY: PEOPLE PIECES 3

Six two-attribute combinations can be made from the four-attribute *People Pieces* set. It will be helpful for you to be aware of the possible combinations as you observe the groupings the students make:

color-girth	color-sex	girth-sex
color-age	girth-age	age-sex

Students will probably arrive at the conclusion that there are six possible two-attribute pairs by making them. Some may approach the problem more analytically. One way of thinking about the task is to recognize that each of the four attributes can be paired with the three other attributes.

color-sex	age-color	sex-age	girth-age
color-age	age-sex	sex-color	girth-color
color-girth	age-girth	sex-girth	girth-sex

There are 4 × 3, or 12, possibilities but because half of them are duplications, there are only six *different* pairings possible.

Younger children will frequently discover two-attribute subsets as they make up stories about the *People Pieces*. Some children may be ready to talk about their stories, and may be pleased to have you listen in, suggest modifications and extensions, or tell stories of your own. If you told a story about the fat children eating lunch together, children might then readily form the group of fat children in which girth and age are the common attributes. Another story might be, "If these people (the fat children) are going to ride in one car, who will ride in the other three cars?" It is not necessary, of course, to have stories for all groupings. If you form one set of pieces that have two attributes in common, children will sometimes spontaneously form the other three sets of pieces which share the same two attributes.

People Pieces 4

Make three-attribute subsets

If the three attributes you have chosen are color, sex, and age, then the pieces in each subset must be *alike* in color, sex, and age. What difference will the pieces in a subset have?

Practice making three-attribute subsets and identifying subsets which your partner has made until you can do it easily. Name the attributes which are shared by the pieces in the subsets as well as the common differences.

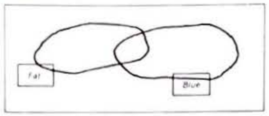
COMMENTARY: PEOPLE PIECES 4

Suppose you place a thin red boy with a thin blue boy and say, "These people are going to walk together." If you then take another piece, a fat red girl perhaps, and say, "Who will walk with this person?" many children will know right away which piece to choose and will form all the other pairs. They will have formed three-attribute subsets by dealing with information given in the first pairing *simultaneously*, not by saying to themselves, "These pairs must be alike in sex, age, and girth." If you or the children have been led by the suggestions on the card to make such a *sequential* analysis of the attributes shared in each pair, it might be useful to do the problem again, making pairs as rapidly as possible without trying to name the attributes. You may sometimes find that you can do this quickly and easily by eye, without using words. It is a common experience, however, that sometimes one's "simultaneous awareness" fails, and sequential analysis is needed to prevent or correct errors. Students working these problems can learn to solve many problems quickly and easily, but they may also discover that there are times when it is important to analyze what has been done, step-by-step.

It is often not obvious, even to students with considerable skill in handling three attributes simultaneously, that it is useful to focus on the common *difference* instead of on the attributes which are shared. Since the pairs will be alike in three out of the four attributes, there will be one attribute which the pairs will not have in common. If a person needs to give himself instructions for making these pairs it may be easier to say, "They must be alike except for sex," than to say, "They must have the same color, the same age, and the same girth." Readiness to consider differences as well as likenesses is useful in a wide range of problem-solving situations.

People Pieces 5

Lay out two overlapping loops. Label one loop with a value of one attribute and the other loop with a value of a different attribute. For example



Can you tell which *People Pieces* will go in each space *before* you put them in?

You will have fat blue people, fat people that are not blue, blue people that are not fat, and people that are neither blue nor fat.

Try this two-loop problem using different values.

Here is a game which you can play with a partner. Choose two label cards and place them blank side up next to the loops. Your partner must discover what the label is for each loop by trying out pieces in any of the spaces. You are allowed to say only "yes" or "no," or "right" or "wrong."

COMMENTARY: PEOPLE PIECES 5

These games are analogous to the ones suggested on Card 29 in the *A Blocks* series. The ideas are presented rather more directly, since students should be able to recognize their similarity to the ideas involved in previous loop games.

In using *People Pieces* with younger children, and perhaps older ones too, it may be helpful to spend some time naming the pieces accurately. For children who can read, it might be interesting to place the value cards as follows:

Red → ***Fat*** → ***Male*** → ***Adult***

⤨ ⤨ ⤨

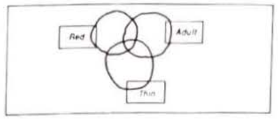
Blue → ***Thin*** → ***Female*** → ***Child***

All sixteen pieces can then be named by reading from left to right, following the various arrows.

Older students may be interested in working out a way of symbolizing the names they have used so that they can keep track of what has been named and what has not. If the values in the top row are represented by 0's and the values in the bottom row by 1's, the number 0001 would represent the red fat male child. A similar use of binary notation is suggested in the Note on page 72.

People Pieces 6

The kind of game suggested on Card 5, involving two loops, can also be played with three loops.



Can you name the pieces which belong in each space without actually putting them there?

Will one of the spaces contain thin, blue adults?

What are the names of the other spaces?

Set up the loops and place the pieces as the labels require.

What happens if you label two of the loops with values of the *same* attribute (for example, *thin* and *fat*)?

Here is a different game. Choose three value cards. They may all be values of different attributes or two of them may be values of the same attribute. Label the three loops with the three value cards, and place the pieces in the appropriate spaces. Turn the cards face down. Your partner now tries to name the cards by looking at the pieces in the loops.

A more challenging version of the game can be played if you label the loops, then have your partner test various pieces to find out where they belong. His task is to identify the labels with as few trials of pieces as possible.

COMMENTARY: PEOPLE PIECES 6

This problem is directly analogous to the one presented on Card 31 of the *A Blocks* series. Students should, with a little practice, be able to name the subsets before placing pieces in the spaces formed by the loops. For the problem suggested, for example, there would be:

thin red adults

thin adults that are not red or
thin blue adults

red adults that are not thin or
fat red adults

adults that are not thin or red or
fat blue adults

red people that are not thin or adult or
fat red children

thin people that are not red or adult or
thin blue children

people that are not thin, red, or adult or
fat blue children

thin red people that are not adult or
thin red children

Since there are only two values of each attribute in the *People Pieces* set, each negative statement can be converted into a positive statement. In the *A Blocks* set this is true only of size, since each of the other attributes has four values. If one of the *People Pieces* is not red, then it is blue; if it is not an adult, then it is a child, and so on. Students who are not confident in handling negative information may find it helpful to make some negative cards, *N-Blue, N-Thin*, etc.

When two loops are labeled by both values of an attribute (adult and child, for example), the intersection will be empty because there are no pieces which are both adult and child.

The problem of determining the labels on the cards which are face down beside the loops is the same as that suggested on Card 35 of the *A Blocks* series. Because there are fewer pieces and because there are only two values of each attribute, students may find that they can handle this problem even if they have had difficulty with the analogous *A Blocks* problem.

People Pieces 7

One of the *People Pieces*

is red
is not fat
is a female
is not a child

Which one is it?

This game can be played in several ways. One player can give the information in the above form or the second player can ask his own questions. It might be interesting if the one who answers responds with all "yes" or all "no" answers, as well as with combinations of the two.

Suppose you want to keep track of the information each question has given you. When you find out that the piece is red you can put all the red pieces together. When you find out that it is not fat you can separate out the thin pieces, etc. Try playing the game this way several times.

Can you play the game by looking at the pieces but not touching them?

Can you play the game with your eyes closed?

COMMENTARY: PEOPLE PIECES 7

This game is similar to the one suggested on *A Blocks* Card 18. Those who have been able to play that one successfully are not apt to have difficulty with this variation.

People Pieces 8

Choose any one of the *People Pieces*. Put this piece, the key piece, on the table. Below it, make a row of all the pieces that differ from it in one way, another row of those pieces that differ from it in two ways, a third row of those that differ from it in three ways, and a fourth row of those that differ from it in four ways.

Each row is a subset of the *People Pieces* set. The subset of pieces that differ in *no* ways from the key piece contains only one piece, the key piece!

How many pieces are there in each of the other subsets?

Can you think of a good way to check whether all your subsets are correct?

COMMENTARY: PEOPLE PIECES 8

This problem generally requires step-by-step analysis; it is easy to get bogged down attempting to solve it all at once.

The analysis is not difficult. There are four pieces which differ in one way from any key pieces, one difference for each of the four attributes:

One Difference color
sex
age
girth

There are six pieces which differ from the key piece in two ways. It may be helpful for students to recall the pairing which they did in the problems suggested on Card 17 of the *A Blocks* series or Card 7 of the *Color Cubes* series. There are six ways of pairing four attributes, just as there are six ways of pairing four blocks.

Two Differences color-sex
color-girth
color-age
sex-girth
sex-age
girth-age

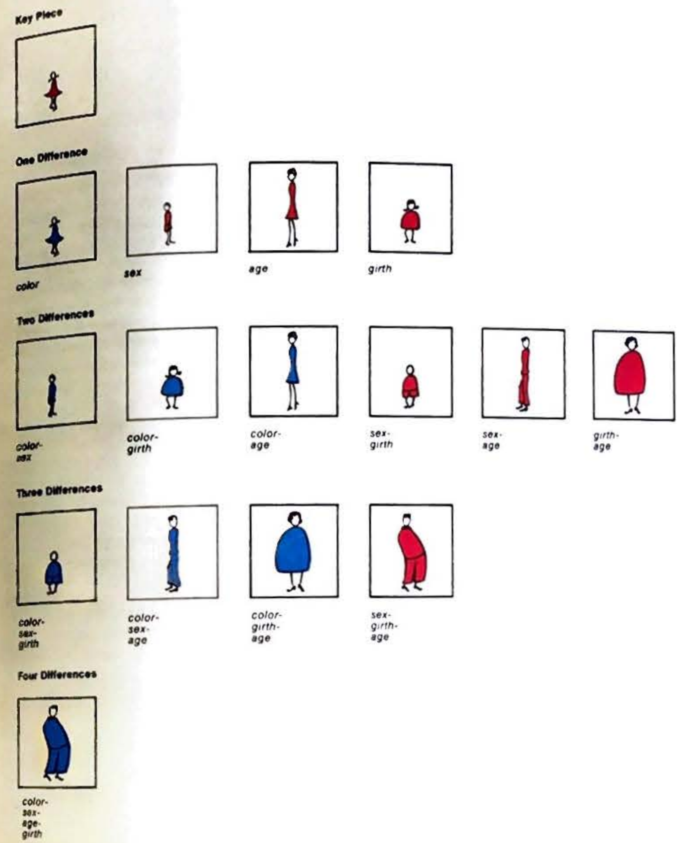
There are four pieces which differ in three attributes from the key piece. One way of thinking about pieces which differ in three ways is to recognize that there will be one attribute in which they are *not* different.

Three Differences color-sex-girth (same age)
color-sex-age (same girth)
color-girth-age (same sex)
sex-girth-age (same color)

There is only one piece which is different from the key piece in all four attributes.

If this problem proves too difficult you can suggest that students try it with half the set: all the reds, or all the males, etc.

Illustration: People Pieces Card 8



People Pieces 9

Choose any two pieces at random and make a pair out of them

How are they different?

How are they alike?

Make other pairs so that the pieces in each pair are alike in the same ways that those in the first pair are. If the pieces in the first pair are alike in age and sex, the pieces in all the other pairs should be alike in age and sex. The pieces in your first pair may be alike in no ways, in one way, in two ways, or in three ways. Practice making pairs until you can do it easily with any number of likenesses.

Pair all the pieces according to a rule which you have chosen. All the pieces in each pair must be alike in three, two, one, or no ways. Place one of the pieces in each pair on top of the other

Can your partner tell, by looking at what is under one piece, what is under all the others? That is, can he discover the rule you have used for making pairs?

Play the game when there are one, two, three, and four differences between the pieces in each pair

If all the pieces are in pairs, according to some rule, can you figure out the rule just by looking at the top pieces, *without* uncovering the bottom piece in one pair first?

People Pieces 10

Take all the adult *People Pieces*

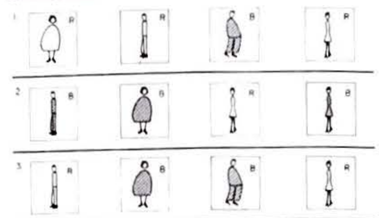
Make a pair with any two pieces

Pair the remaining pieces so that they share the same likenesses and differences. (If your first pair differs just in color, the remaining pairs should differ in color only.)

Place one member of each pair on top of the other

Can your partner tell what is under each one of the cover pieces before he takes any of them off?

Here are three problems involving just the adult pieces. Assuming that they have been paired *systematically*, can you tell what will be under each piece? In one of these groups there is not enough information to solve the problem



Try this with a different subset of eight pieces. You might take all the blues or all the fat people or all the children

Can you predict which problems can be solved and which cannot?

People Pieces 11

Arrange the *People Pieces* in some systematic way

Can you make a matrix out of them?

Try arranging the pieces so that in the rows all the pieces share values of two attributes, for example girth and color, and in the columns they share values of the two other attributes, age and sex

Choose a subset of the pieces which share three attributes

Can these pieces be arranged in a matrix?

Can you make a three-dimensional matrix?

People Pieces 12

Choose any combination of two values. Place the four pieces which share both of these values in a matrix. Place the remaining pieces in groups of four so they correspond to the first matrix

If one matrix is made of fat adults the others will be thin adults, fat children, and thin children

While your partner is not looking, change the position of two of the pieces *within* any one of the subsets, or matrices

Can he find out which pieces you moved and put them back in place?

Again change the position of two pieces while your partner is not looking. This time he should find out which subset you changed and then change pieces in the other three subsets to correspond to the new arrangement you made

When all four subsets are again similar to one another you will have a new starting point for playing the above games again.

Can you think of other kinds of games that can be played starting with an arrangement of four matrices such as that shown above?

COMMENTARY: PEOPLE PIECES 9

Determining the bottom piece in each pair after having seen the bottom piece in one pair requires awareness of the pairing principle. When the pairs have three attributes in common, most students will be able to name the bottom pieces quite easily. The task may be harder when there are only one or two attributes in common or when there are none.

When students are thoroughly familiar with the concept of systematic pairing, they may be able to discover the rule just from looking at the top piece in each pair, without having uncovered any of the bottom pieces. There are some pairings for which this is not possible. For example, if all the red pieces are on top, any of the blue pieces might be under any of the red pieces; that is, all you know for certain about the pairing rule is that it involves a difference in color. On the other hand, if seven red pieces and one blue piece were on top, you would be able to determine the rule, because you would know that a red piece must be under the blue one, and you could determine from looking at the seven visible red pieces which one it must be. If there were six red pieces on top and two blue ones, you could not fully determine what the rule was, but you could narrow the range of possible rules to two.

COMMENTARY: PEOPLE PIECES 10

The first two problems given on Card 10 provide enough information so that it is possible to tell which piece is under each cover if you know that all the pairs share the same likenesses and differences. The first problem has two men and two women, two fat people and two thin people as covers. The key to the solution lies in realizing that one of the covers is blue while the other three are red. Reds *must* be paired with blues because if reds were paired with reds and blues with blues there would be six reds and four blues, whereas you know that in the subset of adults there are four reds and four blues. The piece under the blue cover must be red, and the only red piece that is not visible is the fat red man. Color is therefore the common difference in these pairings.

In the second problem there are three women and one man, three thin people and one fat person, and three blue people and one red person. One can focus on the piece that is different from the other three in any of these ways: the man must be paired with a woman, the fat person with a thin one, the red one with a blue piece, and for each of these pieces there is only one possible choice since the three other possible choices can be ruled out because they are cover pieces. The pairs must therefore differ in color, sex, and girth.

In the final problem there are two men and two women, two fat people and two thin people, two red people and two blue people. Since there is no odd piece, the basis for the pairings cannot be determined without removing one of the covers.

COMMENTARY: PEOPLE PIECES 11

Students who have worked through the *A Blocks* matrix problems should be familiar with this type of activity, and may form a matrix of the *People Pieces* spontaneously. For those who do not, Card 11 suggests some possibilities.

A three-dimensional matrix may prove challenging for anyone. It might be a matrix of four rows, two columns, and two layers. Each row would share common values, as would each column and each pair of pieces in a stack. An arrangement of this sort can be "sliced" so that all the pieces sharing a value of an attribute are together. For example, if all the blue pieces are on top, a horizontal slice will separate the blue and red pieces, a different slice will separate the blue and red pieces, a different slice might separate the adults from the children, or the males from the females. It is not possible, of course, to separate the values of each attribute in a three-dimensional matrix when you are using the full four-attribute set of *People Pieces*. A subset having three common attributes can be arranged so that it is "completely sliceable," however.

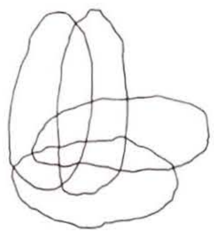
COMMENTARY: PEOPLE PIECES 12

This card presents a problem which is not conceptually difficult, but which can provide a considerable challenge in scanning to find differences. It also suggests a game which can result in systematic transformation of the original matrices into new ones.

Games can be invented which involve transposing pieces among the four matrices. Such transpositions lead to a much greater degree of complexity. It may be best to allow students to discover the possibilities of these *inter-matrix* transpositions for themselves if they are inclined to extend the suggested problems.

People Pieces 13

Lay out four loops like this



Label the first loop with a value of an attribute. Label the second loop with a value of a different attribute. One loop might be *Thin*, another might be *Blue*, etc. Choose values of the other two attributes to label the remaining loops.

Now put each of the *People Pieces* in the appropriate loop or loops.

People Pieces 14

Choose a value, for example, *red*.

Make a line of all the pieces which have this value and then continue the line with all the pieces which have the other value, *blue*. All the reds will be together, followed by all the blues.

Now choose another value, for example, *male*. Within each of the color subsets in your line, the males should come before the females. That is, the first four pieces in the line will be red males, the next four red females, etc.

Decide on a third value. If it is *adult*, then within the subsets of males and females you have already made, the adults will come before the children. Finally, if you choose fatness as the fourth value, the fat red adult male will come before the thin red adult male and so forth.

Starting at one end of the line, stack all sixteen pieces in order. Have your partner try to predict the pieces in the stacking, removing a piece as soon as he has made a prediction about the one under it.

How quickly can he predict correctly?

Can you make up other rules?

Try to invent a way of writing the rules so that your partner can put the pieces in order without talking to you about your rules.

Can you state the rules in words?

COMMENTARY: PEOPLE PIECES 13

This is an extension of the three-loop game with the *People Pieces* which was suggested on Card 6. It may come as a surprise that the *People Pieces* set with four attributes, each having two values, can be placed into the pattern used for the "city-planning" game with the *Color Cubes*. Even more surprising may be the discovery of which piece does *not* go into the pattern. Students may, when they have played this game several times, want to try predicting, for a given combination of values of four attributes, which piece will be left out.

Students can be led by this problem to an explicit awareness that each of the values of the *People Pieces* may be thought of as a negation. For example, if one of the loops is labeled Red, some students may find themselves automatically saying that the pieces that do not go in this loop are the Not-Reds, and then suddenly realize that these are the blues.

The pattern of loops suggested here eliminates the six duplicate spaces of the patterns given on *Color Cubes* Card 1. A different arrangement of loops which eliminates these six duplicate spaces is suggested in the Mathematical Footnote to Color Cubes on page 56.

COMMENTARY: PEOPLE PIECES 14

The problem of ordering this sixteen-piece set in some systematic way opens up a wide range of complexity. Students may find that they can predict the next piece successfully, even though they cannot state the rule. It is also likely that the person doing the ordering will make some errors and this can be the source of profitable discussion.

Note:

There are many ways of describing the different orders, and devising ways of analyzing them can be an interesting puzzle. Here is one way of recording the sequence which was suggested on Card 14.

piece number	*blue*	*old*	*thin*	*male*
1	1	1	1	1
2	1	1	1	0
3	1	1	0	1
4	1	1	0	0
5	1	0	1	1
6	1	0	1	0
7	1	0	0	1
8	1	0	0	0
9	0	1	1	1
10	0	1	1	0
11	0	1	0	1
12	0	1	0	0
13	0	0	1	1
14	0	0	1	0
15	0	0	0	1
16	0	0	0	0

The numeral 1 stands for the presence of the given value; the numeral 0 stands for the presence of its opposite.

Those familiar with binary notation will observe that the order of pieces is given by counting backwards in base two.

People Pieces 15

Latin squares have been introduced on *Color Cubes* Cards 13, 14, 15, and 16 and on *A Blocks* Card 39. You should be familiar with the problems suggested on these cards before attempting a Latin square with the *People Pieces*.

A Latin square can be constructed using all sixteen *People Pieces*. In this case, however, there will be two of each color, two of each sex, two of each girth, and two of each age in each row and in each column.

If you are able to solve this problem, try to recall how you have figured it out. Write down what your reasoning was, or how you went about it, and then compare notes with someone else who has been able to solve it.

Don't be surprised if it takes hours or days for you to solve this. It seems to be the most difficult of the Latin square problems.

People Pieces 16

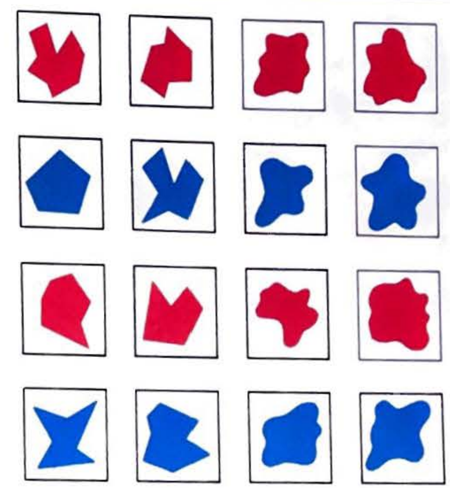
If you have been able to solve the *People Pieces* Latin square problem suggested on Card 15, try mapping it with *Color Cubes*. You will need to establish a code so that a color in a *Color Cubes* Latin square corresponds to two values in the *People Pieces* set. One of the *Color Cubes* Latin squares will be needed for two of the attributes and the other one for the two remaining attributes. One such code might be as follows:

First Latin square (color and sex)	*Second Latin square* (age and girth)
red–red, female	red–adult, fat
blue–blue, female	blue–adult, thin
yellow–red, male	yellow–child, fat
green–blue, male	green–child, thin

If you have been unable to solve the *People Pieces* Latin square, but have found two *Color Cubes* Latin squares which can be placed one on top of the other as suggested in Color Cubes Card 16, you may use these squares to solve the *People Pieces* problem. You may need a partner to help you follow the code that you set up. One cube may tell you, for example, that the piece in the upper left-hand corner has to be red and female while the other cube tells you that it also has to be a thin child. Each pair of *Color Cubes*, therefore, determines four values which can refer to only one of the *People Pieces*.

COMMENTARY: PEOPLE PIECES 15 AND 16

Please see pages 61 and 62 for commentary.

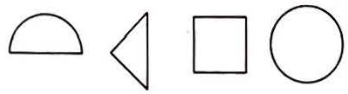


This set of attribute pieces has the same number of attributes and values of each attribute as the *People Pieces* set. Students may enjoy making such sets by drawing on the back of the *People Pieces* set or by drawing the shapes on separate cards. Students will profit from making up attribute sets of their own.

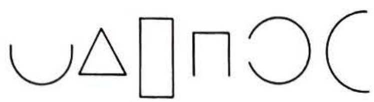
Creature Card 1 *Gligs*

All of these are *Gligs*

None of these is a *Glig*

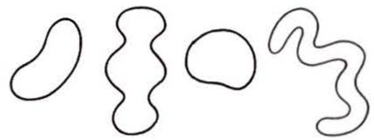


Which of these are *Gligs?*

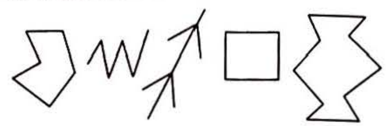


Creature Card 2 *Shlooms*

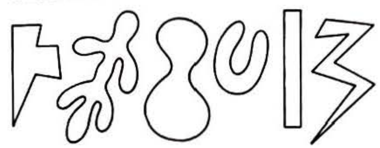
All of these are *Shlooms*



None of these is a *Shloom*

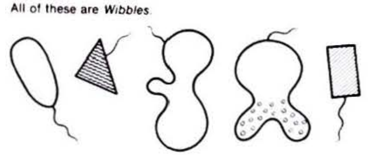


Which of these are *Shlooms?*

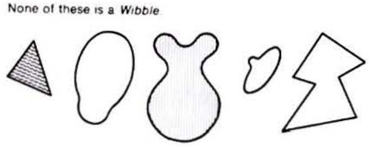


Creature Card 3 *Wibbles*

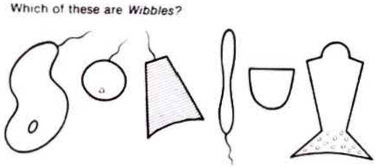
All of these are *Wibbles.*



None of these is a *Wibble.*

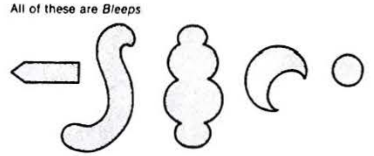


Which of these are *Wibbles?*

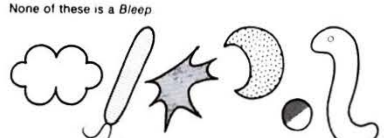


Creature Card 4 *Bleeps*

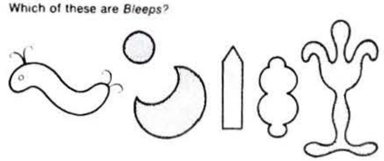
All of these are *Bleeps*



None of these is a *Bleep*

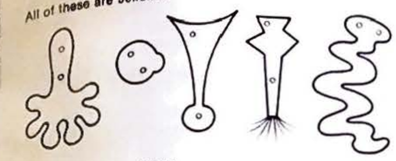


Which of these are *Bleeps?*

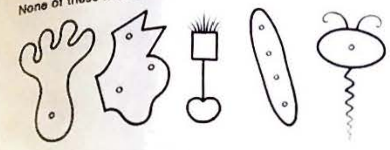


Creature Card 5 *Jexums*

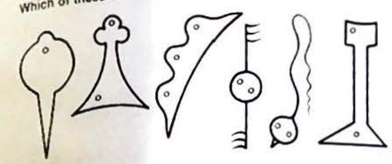
All of these are *Jexums.*



None of these is a *Jexum*

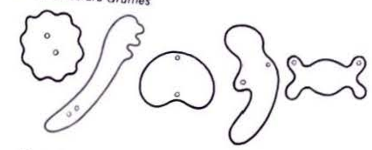


Which of these are *Jexums?*

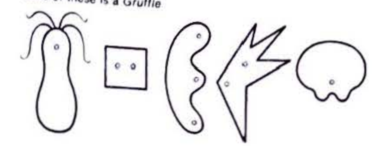


Creature Card 6 *Gruffles*

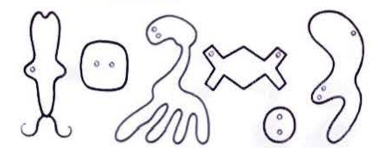
All of these are *Gruffles*



None of these is a *Gruffle*

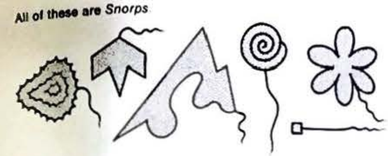


Which of these are *Gruffles?*

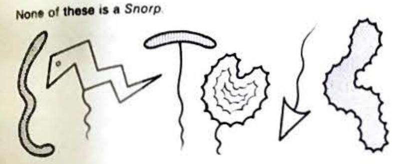


Creature Card 7 *Snorps*

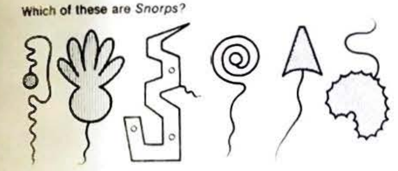
All of these are *Snorps.*



None of these is a *Snorp.*

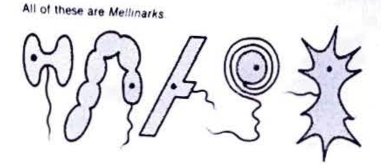


Which of these are *Snorps?*

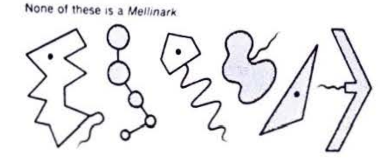


Creature Card 8 *Mellinarks*

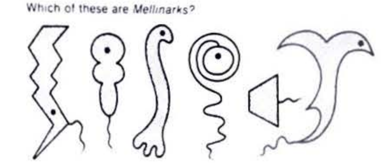
All of these are *Mellinarks.*



None of these is a *Mellinark*

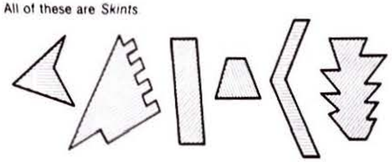


Which of these are *Mellinarks?*

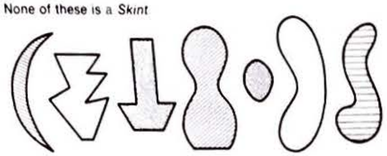


Creature Card 9 *Skints*

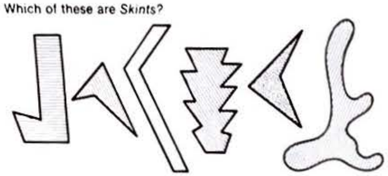
All of these are *Skints*



None of these is a *Skint*



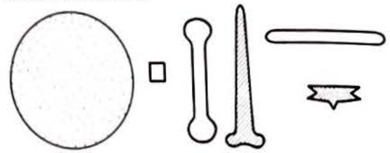
Which of these are *Skints?*



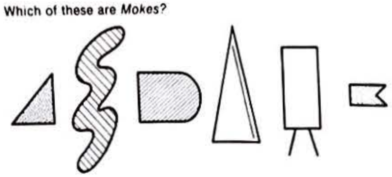
Creature Card 10 *Mokes*

All of these are *Mokes.*

None of these is a *Moke.*

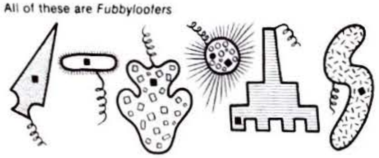


Which of these are *Mokes?*

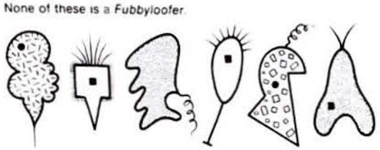


Creature Card 11 *Fubbyloofers*

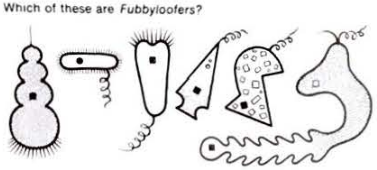
All of these are *Fubbyloofers*



None of these is a *Fubbyloofer.*

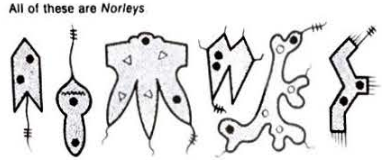


Which of these are *Fubbyloofers?*

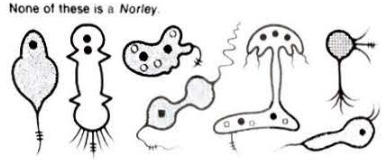


Creature Card 12 *Norleys*

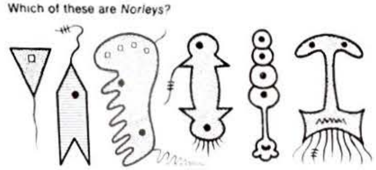
All of these are *Norleys*



None of these is a *Norley.*

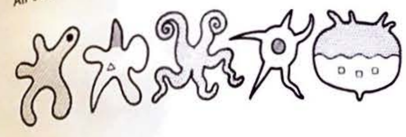


Which of these are *Norleys?*

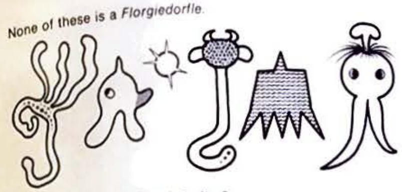


Creature Card 13 *Florgiedorfles*

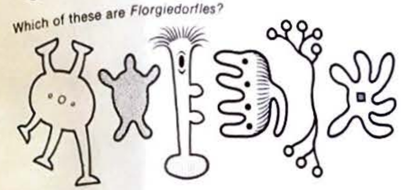
All of these are *Florgiedorfles*



None of these is a *Florgiedorfle.*

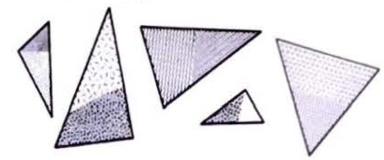


Which of these are *Florgiedorfles?*

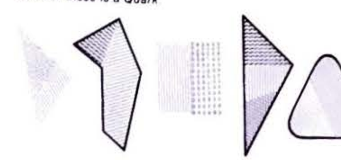


Creature Card 14 *Quarks*

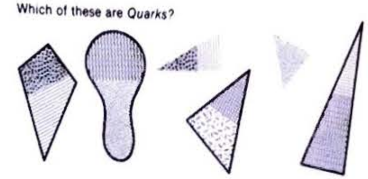
All of these are *Quarks.*



None of these is a *Quark*

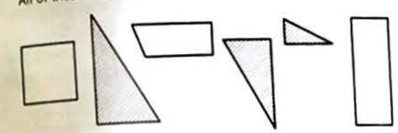


Which of these are *Quarks?*

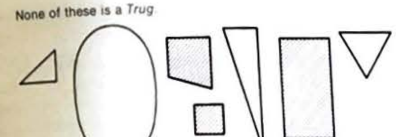


Creature Card 15 *Trugs*

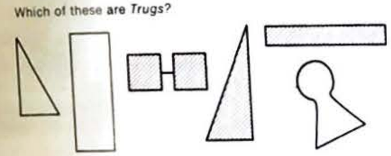
All of these are *Trugs.*



None of these is a *Trug.*



Which of these are *Trugs?*



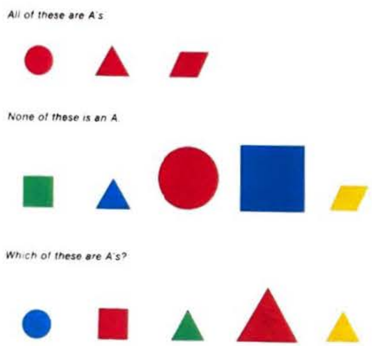
COMMENTARY: CREATURE CARDS

A Blocks, Color Cubes, and *People Pieces* enable students to explore the relationships among attributes which are relatively unambiguous. There is the problem of isolating the attributes of the *A Blocks* and the *People Pieces*, but these attributes remain constant and the emphasis is on their use in forming various subsets. *Creature Cards*, on the other hand, emphasize the identification of defining attributes. It is possible to create, with lines on paper, a wide variety of attributes and combinations of attributes. *Creature Cards* indicate some starting points, a sampling of some of these possibilities. Students should be encouraged to make up their own "creatures" to explore other attributes and combinations.

Each of the first six *Creature Cards* has a single identifying attribute—open figures, curved lines, tails, spots, and so forth. Cards 7, 8, and 9 present creatures which have more than one identifying attribute: Snorps must have spots and tails, Mellinarks must have spots, a black dot, and a tail.

The difficulty of these problems seems to depend upon the number of defining attributes, the amount of irrelevant information or "noise," and on the subtlety of the attribute. Mokes, for example, have a single defining attribute, but many people may not notice at first that all Mokes are the same height. Florgiedorfles, which can be identified by their height and the number of arms, present another difficult problem. Trugs are the only creatures in the series defined by a union instead of an intersection of attributes: shaded triangles or unshaded quadrilaterals are Trugs.

Students may be able to point to the figures in the third row which belong to the class which is named without being able to say how they arrived at their answer. One can have a hunch about a creature



A few cards which are presented in the same format as the Creature Cards may be helpful in getting children to recognize that many of these new problems are basically the same ones they are already familiar with

and be able to solve the problem at a perceptual level, without being able to say why a particular choice was made. The *Creature Cards* can encourage analysis by showing students that they have to check out their hypotheses step by step if they are to avoid errors.

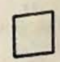
There seems to be a subtle difference between asking a question in the form, "Can you point to the Gligs?" and, "Is this a Glig?" The first form of the question is often easier for younger students who can scan for "Gligness" but have trouble in making a separate yes or no decision for each figure.

Children as young as five have done the first few problems. Our expectation is that children in the early grades can enjoy the first cards if they are presented in a relaxed, casual atmosphere and not in a formal teaching situation.

Children may be interested in comparing the real world with that of the "creature" world. Some five-year-olds tried naming the Wibbles they knew in the real world: dogs, monkeys, horses—anything with a tail. A person with measles or freckles is a Bleep, and there are many two-eyed Jexums around. In comparing creatures in the card set some children observed that all Gruffles are Jexums, but only some Jexums are Gruffles, that all Mellinarks are Snorps, but not all Snorps are Mellinarks.

There will undoubtedly be some disagreement about the creatures that children make up for each other, if not about the cards which are supplied. It may be useful to discuss with children the necessity of having some instances of things which do *not* belong to a class in order to be able to arrive at a definition. If one assumes that any creature can be a member of the class unless there is evidence to the contrary, it is clear that Shlooms can have stripes, tails, eyes, spots, or any other features as long as they have a curved outline. Consider the following sequence in which a very broad definition of class is used.

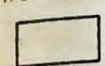
Teacher:
Here is a Quigly:



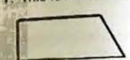
What do you think Quiglies are?

Child:
They're squares.

T: *But this is a Quigly too:*



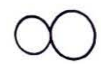
C: *They're any rectangle.*
T: *This is also a Quigly:*



C: *Any four-sided figure is a Quigly.*
T: *This is a Quigly:*

C: *Quiglies have straight lines for sides.*
T: *Here is another Quigly:*

C: *All Quiglies seem to enclose a single space.*
T: *But this is also a Quigly:*



C: *They must be enclosed—no openings.*
T: *This is a Quigly too:*

C: *But everything you have drawn on the paper is a Quigly!*
T: *Yes, a Quigly is anything drawn on paper.*

CHAPTER TWO ATTRIBUTE GAMES AND THINKING SKILLS

Now that you have had firsthand experience with the materials and problems which make up this unit, you may be interested in considering with us the nature of learning and ways in which children may be helped to develop skill and confidence in their own thinking.

A child's habits and styles of thinking and his attitude toward learning are markedly influenced by the conditions under which learning takes place. If speculations about the psychology of learning are to be useful in education, they must take into account a wide range of problems, including the subtle and complex problems of classroom organization. While certain experiences may be more useful than others in helping children develop effective skills of thought, and while it is hoped that the present materials make it possible for many children to have such experiences, it must be pointed out that simply administering a set of exercises to children without taking into account their individual interests and development is unlikely to produce lasting effects. Experiences which result in growth cannot be handed to a child; they are something which he must reach out for and the opportunity for reaching may require conditions quite different from those in which mere passive reacting is sufficient to meet expectations.

Thinking: Out of School and In

Babies and young children play, and we play with them. We do not attempt to teach them in a formal sense, and yet they learn a tremendous amount, largely on their own initiative in an environment of interesting things and responsive people. When they are directly involved in an activity, the attention span of young children is much longer than we normally expect in the classroom, and the intensity of their engagement in the task at hand is often quite astonishing. A child's play is his work, a serious and compelling work for which he uses any materials which may be available. The young child has an active mental life; he seems to be learning necessary and useful things much of the time. In his imagination he creates stories, fantasies which parallel or anticipate the real world.

Language is a most impressive intellectual acquisition, yet children learn to talk without formal instruction. The child does most of the work for himself. At first he pays attention, probably randomly, to many elements in what he hears. He may then begin to pay more selective attention to the elements which seem important because of repetition or of the context in which they occur. He rehearses what he needs to remember, often in a monologue at bedtime. Though learning to talk is an impressive accomplishment, almost all children do learn. If we used traditional practices of formal instruction to teach children to talk, they might never learn. In view of their great potential for learning, any attempt to help children with their thinking must be undertaken with humility, restraint, and respect.

The conditions under which young children learn so much are approximated in some of our nursery schools and kindergartens. In the best of these there is respect for the dignity of the child: his right to learn and his right to choose what he wishes to work on at a particular moment are protected. Under such conditions, the child's capacity for self-direction, his ability to become deeply involved in what he is doing, and his readiness to respond to meaningful interventions by adults and other children, are much greater than is commonly realized.

When a child enters first grade at six, the conditions under which he is expected to learn often change radically. Unfortunately, the pattern in the primary years is increasingly one of formal instruction with most of the choices about content and approach made by the school and the teacher. The teacher, with the aid of the timetable and the curriculum, assumes responsibility for teaching the child and equates teaching with learning—a most dangerous and misleading equation.

Much of the child's potential for learning is lost as soon as someone else attempts to assume responsibility for that learning. The child may or may not be interested in what is taught or in the way it is presented, whereas formerly he was able to teach himself by acting upon his immediate interests. He may learn to pay attention at least part of the time to please the teacher, or to avoid punishment, but he may lose much of his capacity to become deeply absorbed, and his attention span may become a fraction of what it was when he was working on something of his own choosing. The child who can be incredibly persistent in his spontaneous play may find himself in a situation where the only kind of persistence permitted him is that which is applied to an assigned task. If he does become interested in a school activity he must make his interest conform to the schedule of the classroom, putting away the things he is working on and beginning something else many times a day. He learns slowly to conform to the schedule and the discipline of the classroom by setting aside his own interests and learning the game called "school." The kind of self-discipline and personal involvement which enabled the child to acquire language becomes secondary to the discipline of the schoolroom and to the completion of fragments of assigned work.

In addition to limiting scope for initiative and self-direction, formal teaching situations often place other restrictions on children's learning. To enable one teacher to instruct a class, subject matter is broken down into little pieces which are presented one at a time. By checking to see whether these discrete pieces have been retained, the teacher can gain the satisfaction of seeing a measurable result of her teaching. There is growing evidence, however, that it is often more difficult for children to coordinate separate elements presented to them singly than it is for them to deal with much greater complexity which they can handle in their own way. Children may have difficulty understanding things which are presented in little pieces. Drill and repetition can make it appear that they have learned something, but when memory fails, their "learning" disappears. Playing the game of school successfully demands a good memory and an interest in pleasing others. The process of repeating, testing, repeating and testing again, helps the teacher and the school substantiate the illusion that the method is working. Directed teaching of this sort often leads to mediocrity in thinking precisely because the systematic avoidance of challenging complexity makes learning more difficult and less rewarding.

An important idea that has grown out of the investigations that led to this unit is that of "manageable complexity." Many of the games and problems suggested in this unit are complex. The materials available, however, seem to lead most children to realize that although solutions may not be immediately apparent, there is likely to be a way of dealing with the complexity. The materials are rule-bounded: the *A Blocks* set, for example, comprises four shapes, four colors, and two sizes; and once enough pieces are on the table to lead the child to suspect that there are no more shapes, sizes, or colors, he can usually infer what the remaining pieces in the set must be. The city-planning game with *Color Cubes* is a good example of this kind of manageable complexity. There are a number of things for the child to keep in mind as he decides which buildings go in each space, but a visible record of his successful hypotheses is left by the completed buildings, and thus the complexity is not overwhelming. The matrix games suggested on Cards 19 and 20 of the *A Blocks* set are urther examples of manageability: the attributes shared by rows and olumns must be determined in order to solve the problem, but all the iformation needed is present in a form that invites taking steps hich will lead to a solution. Many—though not all—five-year-olds id these problems challenging, as do many—though not all—adults.

eature common to most of the games and problems is that memory iot as important as reasoning ability. In many school settings mory is often crucial, and the child with a good memory may be to succeed with very little real thought or insight. Confronted a new problem or a rearrangement of an old one, however, the who is accustomed to relying on memory may be at a loss. His ing is often as linear as his instruction has been; the steps he ollowed have little relationship to one another; and he may have iotion of where he has been or where he is going. This does not seem to be a necessary pattern for education.

Simultaneous and Sequential Thinking

It may be useful to distinguish between two kinds of thinking, *simultaneous* and *sequential*, always remembering, however, that these terms, or any terms that one might use, are really shorthand for extremely complicated modes of thinking which are imperfectly understood.

In complex situations we often deal with quantities of information virtually simultaneously, without conscious, analytic thought. We perform complex physical acts such as riding a bicycle, driving a car, or swimming, without analyzing the separate components of the acts or being able to explain in words how these components are related. We do the same when we perform complex acts which are not physical. For example, we are all able to recognize people we have seen before without having to stop to consider their individual features.

Consider this last example. It really is remarkable that we are able to recognize hundreds of people all having two eyes, two ears, a nose, and a mouth. Unless an individual has outstanding features, we may find it quite impossible to describe him enough for someone else to recognize him. It is not simply the shape of the nose, the expression of the mouth, or the color of the eyes which triggers recognition. It is, rather, our simultaneous awareness of these and many other cues which gives us a virtually instantaneous impression of a unique individual. We seem to be able to handle a deluge of information at once; our thinking is so swift that we may be completely unaware of how it is proceeding.

Most of the time it is not necessary to analyze the things we do easily and naturally. Analysis becomes important when performance falters or when we wish to become consciously aware of the reasons for the conclusions we reach, or to communicate these reasons to others. Then we must be able to isolate bits of information which we have acted upon and deal with them one at a time, *sequentially*.

We can apparently either deal with many ideas at the same time, unreflectively, or we can separate and analyze the components of a situation, dealing with them one at a time. We probably cannot think in both ways at once, since focusing on isolated aspects of a situation tends to destroy our awareness of the whole. What we may do, however, is to shift attention rapidly back and forth from the whole to its parts, thinking simultaneously or sequentially as required. *Attribute Games and Problems* is designed to help children exercise and become confident of their ability to think in both ways and to employ both modes of thought in complex situations.

Consider your own experience in taking the *A Blocks* out of the box one by one. You were probably able to infer the contents of the set before all the pieces were on the table. You were soon able to tell what was missing without having to consult a list of the pieces. When you built with the pieces and later picked up one of them and named its values, you did so within a framework of awareness of other pieces and their values. In other words, you were able to deal with ideas one at a time or many at a time, and could shift back and forth, easily, between the two kinds of thinking.

A Blocks and the other materials provide the child with an anchor for his thoughts as he shifts from the whole to its parts and back again. The materials are bounded by their defining attributes, which have a high degree of contrast. Bright colors, distinct shapes, and differences in size emphasize the uniqueness of each of the *A Blocks* while also making it possible to group the materials into sets having properties in common. The contrast makes it possible for children to shift from the identification of an individual piece, such as the *small red circle*, to formation of a class, such as *the red pieces*, or one which involves two attributes, such as the *small circles*.

The real world is not as simple as the *A Blocks* world because there are more attributes to deal with, many of them not nearly so easy to distinguish from one another. To become skilled in dealing with this kind of complexity and to develop an awareness of the logical relations between classes of objects, it may be helpful to have had experience with a model comprising a small number of easily identifiable attributes.

The idea of a model is important to an understanding of the possibilities for using the attribute materials. The term is used here in the sense of a simplified mental picture of the important aspects of a complicated problem in the world. We use models continually in everyday life; they correspond to our expectations about situations and objects and might be thought of as maps for maneuvering in a complex environment. We may be unconscious of a model unless we stop to wonder why we were so readily able to deal with a novel set of circumstances.

A Blocks is the one model in this collection. It is supplemented by *People Pieces*, another set of blocks with attributes which are not so easy to name. *Color Cubes* involve six values of only one attribute. *Creature Cards*, drawings on paper, introduce a large variety of values for children to discover. With so many variations, children have many models which may be of use in mastering complex situations.

Classification

Classification plays a major role in our everyday thinking, often without our being aware of it. Forming classes and dealing with their relationships is a part of logic, but we can and do use logic without studying its formal rules. Indeed, logic is of little practical use unless it becomes one part of the way our minds react when confronted with problems. These attribute materials are not intended to teach the formal logic of classification. They are one result of a search to find ways to help children develop skills in thinking which will enable them to use logical relationships in their daily lives. Let us consider the logical as well as the psychological problems involved in categorizing.

Attributes are not tangible objects in the same sense that blocks are, but rather are the properties of objects which we have chosen to isolate and pay attention to. Intellectual development consists, in part, of learning to discover attributes (or categories) relevant to a particular bit of the world and of learning to deal with the relationships among these attributes.

Classification enables us to make sense of a multitude of impressions and to cope with complexity which might otherwise be overwhelming; it gives us tremendous mental powers we would not otherwise possess. It is important to recognize that we have considerable latitude in choosing the attributes we will use in classification. Attributing involves arbitrary decisions to pay attention to certain aspects of experience and not to others.

If you think about the *A Blocks*, you will realize that there are many attributes which could be used to categorize them but which have been ignored in the suggested games. Such attributes might include weight, the material the blocks are made of, surface texture, the sharpness of the corners, and so forth. In their spontaneous play with the pieces, children may pay attention to all these attributes and others which we have not varied systematically. The blocks have been designed so that size, shape, and color are the attributes easiest to use in analyzing the set. If all the blocks were the same color but some weighed one ounce, some two, some four, and some eight ounces, then color would no longer be a particularly useful attribute while weight would be extremely relevant.

The choice of particular attributes for use in analyzing experience restricts the amount of information one will deal with, thus making the universe of objects more manageable. Effective thinking requires skillful choosing. The categories we choose are our responsibility, and we must be flexible and careful in choosing as well as in using them if they are to serve us well.

Undue emphasis on content or concern about the "methodology" of logic is not likely to be useful in helping children to think effectively. Children can learn to talk about sets and subsets, unions, intersection, and negations without having the skill to handle real-life class relations in their heads. It may be unfortunate to force children prematurely into sequential, analytic operations. This may prevent them from exposure to a more challenging complexity than the particular one isolated for analysis and may make it difficult for them to coordinate what they learn in the most useful way.

Summary

For these reasons, this unit emphasizes the importance of free play and the necessity of avoiding sustained periods of directed teaching. Children should be encouraged to invent their own classifications as well as to use conventional ones. The activities suggested in the problem cards represent an attempt to allow children opportunity to develop their talents through exposure to challenging bits of complexity which can be handled by an alternation between simultaneous and sequential functioning, and to become aware of their growing competence as they do so. In free play with the blocks, they may deal with a richness of ideas and associations which is not tapped when the focus is on specific classifying attributes. In classifying and dealing with the relations between classes, on the other hand, they may become aware of a keenness and precision of mind which can be vital in dissecting a complex experience for a specific purpose.

There is considerable evidence that a child's orientation to learning, his habits of thought, become established early in life and are quite persistent. The explorations which children make with attribute materials can provide them with a model which may be useful for thinking in a wide variety of situations, a model which may help them become more sensitive to the world around them, more aware of their own thinking about it, a model which may lead them to an increased confidence in their own intellectual performance.

CHAPTER THREE
USES WITH CHILDREN

The range of complexity in *Attribute Games and Problems* makes it suitable for children from kindergarten age through junior high school —and beyond. Children just entering school will not, of course, approach the materials in exactly the same way as will older children, and the games and problems must be adopted to meet individual situations. It is surprising, however, that some of the skills involved in problem solving of this sort appear to be relatively independent of age, and adults may encounter the same stumbling blocks as five-year-olds. For this reason, it is not possible to prescribe in advance which problems can be undertaken by each grade level. A number of five-year-olds may be able to do some problems with as much skill as most ten-year-olds; within any one class the range of ability to cope with complexity will be great. At the same time, however, younger children may be less able to relate their understanding to other activities in school and in the world outside.

Beyond the broad distinction between self-directed, independent use of materials at the junior high school level and their use in class or in small groups under a teacher's direction at earlier ages, no attempt has been made to specify activities for different age levels. There may, of course, be scope for independent activity on the part of younger children and for class or small group use of the materials by older children. No hard and fast rules can be laid down.

Games and problems should be introduced to younger children (grades K–5) by the teacher. Although it may be possible to present them to the whole class at once, working with smaller groups is usually most effective. While one set of problem cards for the teacher is sufficient, it is advisable to have extra sets on hand for older children who may be ready to proceed independently and who may enjoy the challenge of working from written instructions. The *Color Cubes, People Pieces*, and *Creature Cards* can also be used by pairs of children. Students should have an opportunity to work on the *Creature Cards* individually before sharing their ideas with others. When children have developed considerable familiarity with the *A Blocks*, they can start with the other materials and later return to the *A Blocks* problems.

The amount of material you order for a class will depend on how you plan to use it. It is recommended that this work be done individually or in small groups in kindergarten and the early grades, and in this case two sets may be sufficient for a class. If you wish to make these materials available to more children at the same time, you should order according to the ratio of one set for each pair of children.

A class in which children have a real choice of activities provides the ideal setting for *Attribute Games and Problems*. Here it is possible to introduce the materials in small groups, or even to the entire class, and also make them available for the children to use at will. At this age, children will enjoy playing some of the games that have been suggested by the teacher, making up new games to play with each other, or just exploring the blocks by themselves.

In upper grades where the timetable is flexible, the materials can be introduced and used in much the same way as with younger children. If the classroom schedule is less flexible, the materials can be made available to children to use on their own before classes begin, between classes, during recess, or whenever the situation permits.

Many older children (grades 6–9 and higher) should be able to work directly from the cards, whether the material is used with a class as a whole or as an independent activity by individual children. Many of the problems require a partner, and even some that don't may be most profitably and enjoyably approached when two students are working on them together. For this reason, it is best to have one set of problem cards and materials for each pair of students using the unit.

Students working alone must be free to pursue their own ideas. They should think of the cards as a guide to possible activities, not as a programmed course in logic. Not all students automatically become involved in this way, taking responsibility for their own learning—particularly when they are accustomed to receiving detailed instructions from the teacher. When a student appears to be proceeding too dutifully, following the suggestions on the cards as if they were recipes in a cookbook, it might be advisable to offer the cards as an optional activity and present various problems and games yourself, much as you would with younger children. Students should also be discouraged from racing competitively through the problems.

Some junior high school teachers may wish to use *Attribute Games and Problems* as a supplement to their courses in mathematics or science, to provide more experience in dealing with sets or classification. This is an acceptable approach provided the emphasis remains on developing skills of thought, not on turning the problems and games into a curriculum to be followed in the conventional manner.

Two kinds of learning are required of you as a teacher in using these materials. First, you must explore your own thinking. The insights you gained as you worked through the problems will help you appreciate what the children are doing.

The second kind of learning will come from observing children's thinking as they deal with challenging problems in their own way, from becoming acquainted with the intellectual skills of your students and their potential for growth.

While there are surely ways in which children can be helped to develop more effective skills of thinking, these require a departure from conventional ways of teaching and learning. *Attribute Games and Problems* provides no built-in requirements for study at particular grade levels for prescribed periods of time, nor are there specified goals or rates of advancement. The work should be approached experimentally, in a spirit of adventure, rather than methodically with the expectation that students should be trained up to a certain level of performance that can be carefully measured. There must be a sense of freedom for you and your students. You should not feel that it is your responsibility to get every child to do every problem. Children who are ready can be led on to more challenges, and you in turn should be ready to respond to the challenges individual children set for you and for each other. To insist that all children proceed through the games and problems in the same way and at the same speed would defeat the aim of the entire undertaking. If a student is to develop tenacity, if he is to take a delight in problem solving, he must be free to choose his own problems and his own way of attacking them. He must be free to change the approach, to leave a problem, to return to it, to dwell on one problem for a long time, to reject another.

Although the cards may be used by older students, working independently, they have been provided primarily as starting points for you, as a means of familiarizing you with some of the discoveries children have made and some of the things they have enjoyed doing. You yourself must feel free to be guided by the cards or to put them aside as the occasion requires. Children are far too variable for anyone to be able to provide detailed prescriptions for the way in which this work should proceed.

Learning is a complicated process. To help children you must be ready to stand back, observe what they do, and not succumb to the temptation to give that little bit of assistance which could lead a child prematurely to a solution. You must allow time for your own insights and your awareness of children's potential to develop.

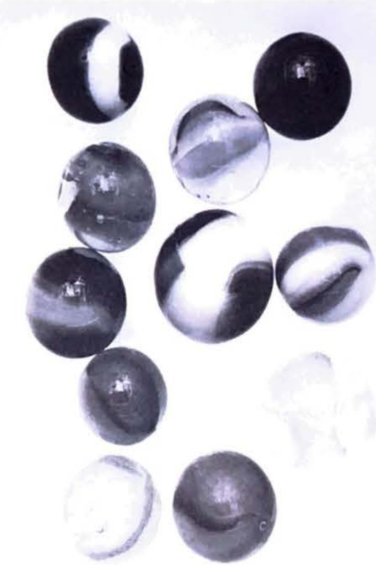
You, the teacher, hold the key to success in using these materials. You must determine the best approach for your class; you should be sensitive to each student's ability and should help him discover a working level high enough to intrigue him but not so high that it will overwhelm him. With experience you will learn when to introduce new problems, when to dwell longer on one problem, when to review, and when to suggest free play.

Children who are familiar with *Attribute Games* should be encouraged to extend them to other materials. Here are some "attribute marbles":

"I am thinking of one of these marbles.
Which one is it?"
(Here are some of the questions children asked.)

"Does it have propellers?"
"Is it large?"
"Is it small?"
"Is it medium-sized?" (The size differences are just noticeable.)
"Does it have spirals?"
"Is it opaque?"
"Is it clear?"
"Is it chewy?" (Some of them have small scratch marks on them.)
"Is it milky?"
"Does it have one color? Two colors? Three colors?"
"Is it pale?"
"Is it bubbly?"
"Is it in the flower family?" (One child noticed that the marbles could **be classified** by the internal pattern.)

An adult asked, "Is it diffuse or discrete?" Indeed, the marbles **could be classified** rather well by this attribute.



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S | science
S | study

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